Horizontal Dirac Operators in CR Geometry

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Introduction

Consider a hypersurface M^{2m+1} of real codimension one in $\mathbb{C}^{m+1}$ or, more generally, in a complex manifold. In particular, such a hypersurface may arise as the boundary of a complex domain $\Omega \subset \mathbb{C}^{m+1}$. Then, its tangent space T_pM at any $p \in M$ cannot be stable under multiplication with the imaginary unit i as it is of odd real dimension. Its largest subspace that is stable under i is given as $H_p = T_pM \cap i \cdot T_pM$. Then, the multiplication with i descends to an isomorphism $J_p \colon H_p \to H_p$ such that $J_p^2 = -\operatorname{Id}_{H_p}$ and the integrability conditions

$$[JX, Y] + [X, JY] \in \Gamma(H)$$
$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0$$

for all $X, Y \in \Gamma(H)$ are satisfied. This is the standard model for CR manifolds and an abstract CR manifold is then defined as an odd-dimensional manifold M^{2m+1} together with a distribution $H \subset TM$ of rank $2m$ and a fibrewise isomorphism $J \colon H \to H$ that satisfy the conditions above. Let it be noted that more general notions of CR manifolds allowing higher codimension of H and not requiring integrability of J exist, but we will content ourselves with the more restrictive definition above.

The oldest results on what has since developed into CR geometry are due to Poincaré [Poi07] who showed that two real hypersurfaces of $\mathbb{C}^2$ are generally not biholomorphically equivalent. This was generalised to higher dimensions by Chern and Moser [CM74] and N. Tanaka [Tan62]. Much of the early literature on CR geometry is decidedly analytic in nature, discussing certain PDEs on CR manifolds, for instance [Koh63, KR65, Koh65].

An more geometrically flavoured investigation began with the works of S.M. Webster [Web78] and N. Tanaka [Tan75] introducing a metric and a connection on CR-manifolds as follows: If M is oriented, then there exists a (non-unique) one-form $\eta \in \Omega^1(M)$ such that $H = \ker \eta$. To such a form, one associates the Levi form $L_\eta \in \Gamma(\operatorname{Sym}^2 H^*)$ via

$$L_\eta(X, Y) = \tfrac{1}{2} d\eta(X, JY).$$

If L_η is positive-definite, we obtain a metric g_η on M, the so-called Webster metric defined by

$$g_\eta = L_\eta + \eta \otimes \eta,$$

which allows us to do geometry in the Riemannian sense on M. In this case, (M, H, J, η) is called a strictly pseudoconvex CR manifold. On such a manifold, η is in fact a contact form, i.e. it satisfies $\eta \wedge (d\eta)^m \neq 0$, and moreover, g_η is compatible with η and J in the sense that $2g_\eta(JX, Y) = d\eta(X, Y)$, where we extend J by zero on $H^\perp$. This includes strictly pseudoconvex CR manifolds in the larger class of metric contact manifolds, i.e. tuples (M, g, η, J) of a Riemannian metric g on M, $\eta \in \Omega^1(M)$ and $J \in \operatorname{End}(M)$ such that $\|\eta\|_g = 1$ (pointwise), $2g(JX, Y) = d\eta(X, Y)$ and $J^2 = -\operatorname{Id} + \eta \otimes \eta^\sharp$. Note that the Riemannian structure on a CR manifold is unique only up to the choice of η which can be conformally changed by any function $u \in C^\infty(M)$ to obtain a new strictly pseudoconvex structure form $\tilde{\eta} = e^{2u}\eta$.

The main ingredient in Riemannian geometry beside a metric g is a connection ∇. In particular, if one has additional geometric objects on the manifold, one would like them to be parallel under the connection. In the case of a strictly pseudoconvex CR manifold, J will never be parallel under the Levi-Civita connection ∇^g (unlike in the almost-Hermitian case, where J is parallel under ∇^g in the subclass of Kähler manifolds). Thus, we replace ∇^g with an adapted connection, i.e. a metric connection that parallelises J (and thus η). The most well-known of these adapted connections is the Tanaka-Webster ∇^η connection developed independently by N. Tanaka and S.M. Webster that is defined through its torsion, which is given by

$$T(X, Y) = d\eta(X, Y)\xi \quad \text{and} \quad T(\xi, X) = \tfrac{1}{2}([X, \xi] + J[JX, \xi])$$

for $X, Y \in \Gamma(H)$, where ξ is the Reeb vector field characterised equivalently as the metric dual of η or by $\eta(\xi) = 1$ and $\xi \lrcorner d\eta = 0$.

While the Tanaka-Webster connection is used in most work on CR manifolds, the space of adapted connections is much larger. Some examples of adapted connections have been constructed by L. Nicolaescu [Nic05] (through Hermitian connections on $M \times \mathbb{R}$) and (in the five-dimensional case) by C. Puhle [Puh11]. It is then a natural question how large the space of adapted connections is and how they can be characterised. In the almost-Hermitian case (which, to some extent, is the even-dimensional analogue of metric contact manifolds) such a classification of Hermitian connections is available, it is due to P. Libermann [Lib54] (compare also the discussion in modern language by P. Gauduchon [Gau97]). In this thesis, we give an explicit description of the class of adapted connections on a general metric contact manifold. Any metric connection is defined by its torsion tensor and through a careful decomposition of the space $\Omega^2(M, TM)$, we can explicitly describe the space of torsion tensors of adapted connections (cf Theorem 1.4.3).

Theorem. *Let (M, g, η, J) be a metric contact manifold and ∇ an adapted connection. Then its torsion tensor has the following form:*

$$T = N^{0,2} + \tfrac{9}{8}\omega - \tfrac{3}{8}\mathfrak{M}\omega + B + \xi \otimes d\eta - \tfrac{1}{2}\eta \wedge (J\mathcal{J}) + \eta \wedge \Phi,$$

where ω is a three-form on H whose decomposition into (p,q)-forms consists only of forms of type $(2,1)$ and $(1,2)$, $B \in \Omega^2(H,H)$ satisfies $B(J\cdot, J\cdot) = B$ and vanishes under the Bianchi operator and Φ is a skew-symmetric endomorphism of H satisfying $\Phi J = J\Phi$. The other parts are completely determined by the geometry of the contact structure.

Conversely, given any ω, B, Φ as above there exists exactly one adapted connection $\nabla(\omega, B, \Phi)$ whose torsion is as given above.

In this description, the Tanaka-Webster connection is the connection obtained by setting all freely choosable parts to zero.

We would now like to do spin geometry over CR manifolds. In Riemannian manifolds, a lot of geometric information is reflected in the spin Dirac operator of the manifold and the hope is that the CR geometry is similarly reflected in an appropriately chosen Dirac operator. Beside this interest for the possible information about the CR geometry itself, this is also of interest in Lorentzian geometry. Through the Feffermann construction (originally described by C. Feffermann [Fef76] for products $\partial\Omega \times S^1$ where $\Omega \subset \mathbb{C}^{m+1}$ is a complex domain), one obtains a Lorentzian metric F_η on S^1-bundles over a strictly pseudoconvex CR manifold (M, H, J, η). Under a conformal change of the contact form η on M, F_η is also changed conformally. H.Baum [Bau99] used the relation between the CR geometry and Lorentzian geometry to construct twistor spinors on Feffermann spaces.

A spin structure on an oriented Riemannian manifold (M^n, g) is a reduction of the bundle of orthonormal frames $P_{SO}(M)$ to a bundle $P_{Spin}(M)$ with the spin group $Spin_n$ as the structure group. There are topological obstructions to the existence of a spin structure, but many oriented manifolds do admit one. The spin group has a canonical representation $\kappa\colon Spin_n \to \Delta_n$ on the spinor module $\Delta_n \simeq \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ and we thus obtain a vector bundle $\mathbb{S} = P_{Spin}(M) \times_\kappa \Delta_n$. This vector bundle carries a Hermitian bundle metric and a Clifford multiplication structure, i.e. there exists a (fibrewise) multiplication of tangent vectors of M with spinors (sections of $\mathbb{S}$) $cl\colon TM \times \mathbb{S} \to \mathbb{S}$.

Any metric connection on M induces a connection on $\mathbb{S}$ that is metric with respect to the Hermitian metric. This furthermore induces a first-order differential operator on M via

$$D^\nabla\colon \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathbb{S}) \overset{g}{\simeq} \Gamma(TM \otimes \mathbb{S}) \xrightarrow{cl} \Gamma(\mathbb{S}).$$

This operator is called the *Dirac operator* associated with ∇. Beyond motivation from the Dirac equation in physics, the Dirac operator (for ∇^g) rose to mathematical prominence due to its role in the works of Atiyah and Singer on index theory.

While there were some results exploring Dirac operators associated with different connections (see [FS79]), attention quickly focused on the case $\nabla = \nabla^g$. In this case, the Dirac operator is a formally self-adjoint elliptic differential operator. If (M, g) is complete, it is essentially self-adjoint and its spectrum on a closed manifold is a discrete pure point spectrum tending to infinity. This spectrum has received a lot of attention. It is notoriously difficult to calculate the spectrum explicitly and only a few examples are know, mostly homogeneous manifolds, where one can use representation theory to calculate the spectrum. Lower bounds for the lowest eigenvalue λ_0 have instead been produced, starting with Friedrich's estimate [Fri80]: $\lambda_0^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{scal}^g$. This lower bound is attained on spheres and is thus optimal. The estimate can, however, be improved on manifolds admitting (∇^g-) parallel structures, for instance Kähler (as proven by K. Kirchberg [Kir86]) and Quaternion Kähler (as shown by W. Kramer, U. Semmelmann and G. Weingart [KSW99]) manifolds. A survey of results on the spectrum of the Dirac operator can be found in the book by Ginoux [Gin09].

As the relevant structures on a CR manifold are not parallel under the Levi-Civita connection, it is not reasonable to expect the geometry of such a manifold to be reflected in D^{∇^g}. The investigation of Dirac operators associated with other connections is much younger and has so far mostly been focused on connections with skew-symmetric torsion, see for instance [FI02, AF04, AFK08].The adapted connections on CR (or metric contact) manifolds do not generally fall into this category. In fact, only a single adapted connection with skew-symmetric torsion exists on a subclass of CR manifolds.

We discuss Dirac operators associated with adapted connections in this thesis. They are still elliptic differential operators, but their self-adjointness depends on the connection. In fact, it is well known that the Dirac operator D^{∇} is formally self-adjoint if and only if $\mathrm{div}^{\nabla} = \mathrm{div}^g$ (cf Proposition 2.3.3). For an adapted connection $\nabla = \nabla(\omega, B, \Phi)$ this is equivalent to $\mathrm{tr}(B) = \frac{3}{8} \mathrm{tr}\, \mathfrak{M}\omega$. Moreover, if (M, g) is complete, the Dirac operator of any such connection is essentially self-adjoint.

However, these Dirac operators are not well-adapted to the geometry of a CR manifold either. Recall that the Riemannian structure of a CR manifold depends on the choice of the contact form η, which can be replaced with $e^{2u}\eta$. Such a change induces a conformal change of $L_\eta = g_\eta|_H$, but the change for the whole metric g_η (and also the change of the Reeb vector field) is much more involved. Consequently, the Dirac operator D^{∇^η} does not transform nicely under such a change. It therefore makes more sense to focus on the Sub-Riemannian structure (H, L_η). This has for instance been done in the study of the Laplacian on CR manifolds, which is replaced with the horizontal Laplacian that derives only in the direction of H. For this Laplacian, lower bounds for its spectrum [Gre85, BD97] and Obata-type theorems [IV12, LW13] have been produced.

In analogy with this, we consider the horizontal Dirac operator

$$D_H^{\nabla} \colon \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathbb{S}) \overset{g}{\simeq} \Gamma(TM \otimes \mathbb{S}) \xrightarrow{\mathrm{pr}_H \otimes \mathrm{Id}} \Gamma(H \otimes \mathbb{S}) \xrightarrow{cl} \Gamma(\mathbb{S}).$$

This operator is indeed well-adapted to the Sub-Riemannian structure as the following result shows (cf Theorem 3.3.3).

Theorem. *Let (M^{2m+1}, H, J, η) be a spin strictly pseudoconvex CR manifold and $\tilde{\eta} = e^{2u}\eta$. Then, the horizontal Dirac operator associated with ∇^η and $\widetilde{\nabla^\eta} = \nabla^{\tilde{\eta}}$ transforms as follows:*

$$\widetilde{D_H^{\nabla^\eta}\varphi} = e^{\frac{2m+1}{2}u} D_H^{\widetilde{\nabla^\eta}} \left(e^{-\frac{2m-1}{2}u}\tilde{\varphi}\right),$$

where $\tilde{}$ denotes the isomorphism between the spinor bundles of (M, g_η) and $(M, g_{\tilde{\eta}})$.

Horizontal Dirac operators have been discussed but in few articles to date: R. Petit [Pet05] considered the horizontal Dirac operator of $\nabla = \nabla^\eta$ and produced Schrödinger-Lichnerowicz type formulae and vanishing theorems. I. Kath and O. Ungermann [KU13] and S. Hasselmann [Has14] investigated horizontal Dirac operators associated with Sub-Riemannian geometry. Kath and Ungermann focused on quotients of nilpotent groups, provided a general framework to calculate the spectrum of D_H^∇ on such manifolds and explicitly calculated the spectrum in some examples. Hasselmann focused on Carnot groups and the analytic properties of D_H^∇. A large part of this thesis is devoted to a systematic discussion of these operators.

The horizontal Dirac operators mirror many properties of the full Dirac operators, replacing some geometric objects with their horizontal counterparts where necessary. They behaves on products of functions and spinors (Lemma 3.1.2) as well as vector fields and spinors (Lemma 3.1.5) in a similar way as the full operator. The criterion for self-adjointness is the same as for the full Dirac operator (cf Proposition 3.1.1).

Proposition. *Let (M, g, J, η) be a spin metric contact manifold and ∇ an adapted connection on M with torsion T. Then, the horizontal Dirac operator D_H^∇ is formally self-adjoint if and only if the full Dirac operator D^∇ is formally self-adjoint. This is the case if and only if the torsion tensor is traceless: $\operatorname{tr} T = 0$.*

In particular, the operator D_H^η associated with the Tanaka-Webster connection is formally self-adjoint.

The square of a horizontal Dirac operator is of what one might call Sublaplace type and admits a Weitzenböck-type formula (Theorem 3.2.4).

Theorem. *Let (M, g, η, J) be a closed spin metric contact manifold and ∇ an adapted connection with traceless torsion T, i.e. whose horizontal Dirac operator D_H^∇ is formally self-adjoint. Then, there exists a connection ∇^W on $\Gamma(\mathbb{S})$ and an endomorphism E of $\mathbb{S}$ such that*

$$(D_H^\nabla)^2 = (\nabla_H^W)^* \circ \nabla_H^W - d\eta \cdot \nabla_\xi + E, \tag{1}$$

where $\nabla_H \colon \Gamma(\mathbb{S}) \to \Gamma(H^* \otimes \mathbb{S})$ is the restriction of ∇ and the adjoint is taken with respect to the L^2-inner product.

If $\nabla = \nabla(\omega, B, \Phi)$, then ∇^W is adapted if and only if the manifold is CR and $\omega = 0$.

The first-order term ∇_ξ cannot be avoided here as the product of the horizontal operator D_H^∇ has contributions of lower order in the transversal direction, whereas the horizontal connection Laplacian $(\nabla_H)^* \circ \nabla_H$ does not – the latter fact is independent of the connection ∇.

If we focus on the Tanaka-Webster connection (which takes the role of ∇^g as the canonical choice of connection in CR geometry), we can refine some of the above results and obtain additional properties of the formally self-adjoint operator $D_H^\eta = D_H^{\nabla^\eta}$, which we will call the Tanaka-Webster operator. For this operator, the Weitzenböck formula above simplifies to a Schrödinger -Lichnerowicz type formula where $\nabla^W = \nabla^\eta$ and the endomorphism E is simply multiplication with the scalar curvature (cf Theorem 3.2.5). To get rid of the first-order part $d\eta \cdot \nabla_\xi^\eta$, one can replace the horizontal connection Laplacian by the connection Laplacians $(\nabla^\eta_{(1,0)})^*(\nabla^\eta_{(1,0)})$ and $(\nabla^\eta_{(0,1)})^*(\nabla^\eta_{(0,1)})$ to obtain another Schrödinger-Lichnerowicz type formula (Theorem 3.2.8), where $\nabla^\eta_{(1,0)}$ and $\nabla^\eta_{(0,1)}$ are appropriate restrictions of the $\mathbb{C}$-linear extensions of ∇^η to arguments form $H \otimes \mathbb{C}$. Both Schrödinger-Lichnerowicz type formulae are originally due to Petit [Pet05].

The spinor bundle over a CR manifold splits as

$$\mathbb{S} = \bigoplus_{k=0}^{m} \mathbb{S}_{m-2k}, \quad \text{where } d\eta \cdot \varphi = 2i(m-2k)\varphi \text{ for } \varphi \in \mathbb{S}_{m-2k}.$$

While D_H^η does not conserve this splitting but rather

$$D_H^\eta(\Gamma(\mathbb{S}_{m-2k})) \subset \Gamma(\mathbb{S}_{m-2(k-1)} \oplus \mathbb{S}_{m-2(k+1)}),$$

the square $(D_H^\eta)^2$ does map $\Gamma(\mathbb{S}_{m-2k})$ to itself.

The operators D_H^∇ are not elliptic anymore. The question thus arises whether they still have “nice” analytic properties like the elliptic operators. In particular, among these “nice” properties that we would like to have are hypoellipticity (or regularity, the property that if we extend an operator to distributions, then if Pu is smooth, so is u) and the discrete pure point spectrum on closed manifolds. The appropriate analytic theory to investigate these questions is the Heisenberg calculus. This is a symbolic calculus for operators whose highest-order parts are (typically) in the direction of a codimension one subbundle $H \subset TM$ and the contributions in the transversal direction are of lower order.

In the Heisenberg calculus, the tangent space is replaced by a two-step nilpotent tangent group $T_H M_a$. The underlying vector space of its Lie algebra $\mathfrak{t}_H\mathfrak{m}_a$ is $H_a \oplus (TM_a/H_a)$ and therefore, any vector field $Y \in \mathfrak{X}(M)$ defines an element of $\mathfrak{t}_H\mathfrak{m}_a$

and thus, a left-invariant vector field Y^a on $T_H M_a$. Then, a differential operator P that is locally of the form

$$P|_U = \sum_{\langle\gamma\rangle \le k} a_\gamma X^\gamma,$$

where $X = (X_0, \ldots, X_{2m})$, $X_1, \ldots, X_{2m}$ span H_U, X_0 is transversal and $\langle\gamma\rangle = 2\gamma_0 + \gamma_1 + \cdots + \gamma_{2m}$, induces a left-invariant homogeneous differential operator

$$P^a = \sum_{\langle\gamma\rangle = m} a_\gamma(a)(X^a)^\gamma$$

on $T_H M_a$. This operator is called the model operator of P at the point a. The differential operator P is hypoelliptic if and only if P^a is at every point. The left-invariant homogeneous operator P^a is hypoelliptic if and only if $\pi_*(P^a)$ is injective for any nontrivial unitary representation π of $T_H M_a$, where π_* is the induced representation on $\mathfrak{t}_H\mathfrak{m}_a$ with the obvious extension to operators. This condition is called the Rockland condition (Theorems 4.5.2 and 4.5.4). Any differential operator satisfying this criterion will have analytic properties that are essentially those we know from elliptic operators (Theorem 4.5.14).

For a strictly pseudoconvex CR manifold M^{2m+1} and a differential operator of Sublaplace type, i.e. an operator of the form

$$-\sum_{j=1}^{2m} (X_j)^2 + \mu(a) X_0 + \sum_{j=1}^{m} b_j X_j + E,$$

where E is the part of order zero and $(X_0, \ldots, X_{2m})$ as before, the Rockland condition is equivalent to the condition

$$\operatorname{spec} \mu(a) \cap \left\{ \pm 2 \left(m + \sum_{j=1}^{m} \nu_j \right) \,\middle|\, \nu_j \in \mathbb{N}_0 \right\} = \emptyset.$$

The square of a horizontal Dirac operator D_H^∇ is of Sublaplace type. Applying the above condition, we find that independently of ∇, D_H^∇ is not hypoelliptic. In the case of the Tanaka-Webster operator, this result can be refined, however. The restrictions of $(D_H^\nabla)^2$ to the bundles $\mathbb{S}_{m-2k}$ for $k \neq 0, m$ are hypoelliptic and moreover, using the hypoellipticity on these parts, one can obtain analytic properties on the "extremal" bundles as well (Proposition 4.5.16 and Theorem 4.5.19).

Proposition. *Let (M, H, J, η) be a closed spin strictly pseudoconvex CR manifold of dimension $2m+1 \ge 5$ and D_H^η the horizontal Dirac operator induced by its Tanaka-Webster connection. Then, $(D_H^\eta)^2$ has pure point spectrum, the eigenvalues are real, nonnegative and tend to infinity. The eigenspaces associated with the nonzero eigenvalues are finite dimensional and consist of smooth sections of the spinor bundle. The same holds for* $\ker((D_H^\eta)^2) \cap L^2(\mathbb{S}_{m-2k}) = \ker((D_H^\eta)^2) \cap \Gamma(\mathbb{S}_{m-2k})$ *for $k \neq 0, m$.*

Calculating the full spectrum is – unsurprisingly – no easier than in the case of full Dirac operators, but there are again examples where this is possible, often building on the techniques used for D^{∇^g}. One of these examples are S^1 bundles $\pi\colon M \to \bar{M}$, where the close relationship between the CR geometry of the total space and the Kähler geometry of the base space allows us to compare ∇^η and $\nabla^{\bar{g}}$ and obtain

$$D^\eta_H(\varphi \circ \pi) = (D^{\bar{g}}\varphi) \circ \pi \quad \text{for any } \varphi \in \Gamma(\bar{\mathbb{S}}).$$

This allows us to calculate part of the spectrum for spheres, which are S^1-bundles over $\mathbb{C}P^n$ (Corollary 3.4.6 and Theorem 3.4.9).

Theorem. *The values $\pm\sqrt{\lambda_{a,b}}$ and $\pm\sqrt{\mu_{a,b}}$ are contained in the point spectrum of the Tanaka-Webster operator of the sphere S^{4k+3} with the standard Sasaki structure, where:*

$$\begin{aligned}
\lambda_{a,b} &= (a+k)(a+2k+1-b)\\
&\quad b \in \{1,...,2k+1\} \text{ and } a \geq \max\{1, b-k\}\\
\mu_{a,b} &= (a+k+1)(a+2k+1-b)\\
&\quad b \in \{0,...,2k\} \text{ and } a \geq \max\{0, b-k\}.
\end{aligned}$$

the associated eigenspinors are all of type $\varphi \circ \pi$, where $\pi\colon S^{4k+3} \to \mathbb{C}P^{2k+1}$ is the Hopf fibration and φ is an eigenspinor of the Dirac operator associated with $\nabla^{\bar{g}}$ on $\mathbb{C}P^{2k+1}$.

On S^3 equipped with its standard Riemannian and CR structure, the following values are eigenvalues of $(D^\eta_H)^2$.

$$\begin{aligned}
\lambda_a &= a^2 && a \in \mathbb{N}_0\\
\lambda^+_{p,q} &= 4pq && p, q \in \mathbb{N}_0, p+q \neq 0\\
\lambda^-_{p,q} &= 4(1+pq+p+q) && p, q \in \mathbb{N}_0.
\end{aligned}$$

The eigenspinors associated with the eigenvalues λ_a are of the same type as above, whereas the eigenspinors associated with the remaining eigenvalues are not lifts of a spinor on $\mathbb{C}P^1$.

The other class of examples are homogeneous manifolds. While the general approach for calculating the spectrum of D^{∇^g} does not readily carry over to other connections, some particular examples are accessible using representation techniques. If we consider a discrete subgroup $\Gamma < G$ of a nilpotent Lie group with a left-invariant CR structure, then the general theory is not necessary. This approach has been discussed in [KU13] and the three-dimensional Heisenberg group was discussed as one example. This approach has also been used in [Has14] for products of the Heisenberg group and euclidean space (of any dimension), where the spectrum of $(D_H)^2$ was determined. For dimensions three and five, the spectrum of the Tanaka-Webster operator on compact quotients of the Heisenberg group has the following form (cf Theorems 3.5.1 and 3.5.4), see also [KU13, section 3.4] and [Has14, Theorem 4.3.4].

Theorem. *Let $\mathcal{H}^m$ $(m = 1, 2)$ be the Heisenberg group of dimension $2m + 1$ and*

$$\Gamma_r = \left\{ (rx, y, z) \,\middle|\, (x, y, z) \in \mathbb{Z}^{2m+1} \right\} < \mathcal{H}^1 \quad (r \in \mathbb{N}^m, r_j \text{ divides } r_{j+1})$$

be a lattice. On the quotient manifold $H = \Gamma_r \backslash \mathcal{H}^m$, let the spin structure be defined by the homomorphism

$$\varepsilon \colon \Gamma_r \to \mathbb{Z}_2, \quad \varepsilon(rx, y, z) = \delta_1^{x_1} \delta_2^{y_1} \cdots \delta_{2m-1}^{x_m} \delta_{2m}^{y_m} \delta_{2m+1}^{z},$$

where $\delta_1, \ldots, \delta_{2m+1} \in \{\pm 1\}$ and $\delta_{2m+1} = 1$ if r_j is odd for some j.

Then, the eigenvalues of D_H^η are given as follows:

In the case $\delta_{2m+1} = 1$, D_H^η has an infinite-dimensional kernel and the following nonzero eigenvalues:

$$\begin{aligned} \lambda_\beta^\pm &= \pm 2\pi \sqrt{\|\beta\|} && \beta \in B, \\ \lambda_{\alpha,k}^\pm &= \pm 2 \sqrt{\pi \alpha (k_1 + \cdots + k_m)} && \alpha \in \mathbb{N}, k \in (\mathbb{N}_0)^m, \end{aligned}$$

where

$$B = \left\{ \beta \in (\tfrac{1}{2r}\mathbb{Z} \times \tfrac{1}{2}\mathbb{Z})^m \,\middle|\, e^{2\pi i r \beta_{2j-1}} = \delta_{2j-1}, e^{2\pi i \beta_{2j}} = \delta_{2j} \right\}.$$

The eigenvalues have the following multiplicities: The multiplicity of $\lambda_\beta^\pm$ has multiplicity 1 for each admissible β and $\lambda_{\alpha,k}^\pm$ has multiplicity $2\alpha r_1 \cdots r_m$.

In the case $\delta_{2m+1} = -1$, D_H^η has an infinite-dimensional kernel and the following nonzero eigenvalues:

$$\lambda_{\alpha,k}^\pm = \pm 2 \sqrt{\pi \alpha (k_1 + \cdots + k_m)} \qquad \alpha \in (\mathbb{N}_0 + \tfrac{1}{2}), k \in (\mathbb{N}_0)^m.$$

which have multiplicity $2\alpha r_1 \cdots r_m$.

Structure of this thesis

In the first chapter, we introduce CR and contact metric manifolds and review some of their basic properties that will be use in the rest of the thesis. We then carefully study the space $\Omega^2(M, TM)$ to obtain a characterisation of the adapted connections and give some applications of this result.

The second chapter is devoted to spin geometry. We first review the basic theory of Clifford algebras and spin representations as well as spinor bundles over manifolds and their connections and Dirac operators before focusing on contact metric and CR manifolds and the Dirac operators associated with adapted connections.

The third chapter is where we finally come to horizontal Dirac operators. We give a systematic exposition of their properties, including both known and new results. As examples, we consider the Tanaka-Webster operator D_H^η on S^1-bundles and homogeneous manifolds.

In the fourth and final chapter, we give an introduction to the Heisenberg calculus that is hopefully more accessible to a differential geometer than the original papers. We then apply the hypoellipticity criterion from the Heisenberg calculus to horizontal Dirac operators, with particular attention on the Tanaka-Webster operator. Some of the facts from functional analysis used in this chapter are collected in an appendix.

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1 Contact and CR manifolds and their adapted connections

In this chapter we review the basic objects that underlie our research: Contact and CR manifolds and adapted connections on these manifolds. In the first two sections we introduce contact manifolds, with a focus on metric contact manifolds, and CR manifolds, with an emphasis on strictly pseudoconvex CR manifolds. We then discuss adapted connections on these manifolds, i.e. connections that parallelise all relevant data, and present a way to fully describe all possible such connections through their torsion.

1.1 Contact manifolds

Essentially, a contact structure on a manifolds is given by a hyperplane distribution that is "as far from being integrable as possible". We will quickly review these structures, before moving on to metric contact manifolds, which, through their metric, allow us to do geometry on them. The material of this section is mostly well-known and the interested reader can find more details from a geometric viewpoint in the book by Blair [Bla02] and a more topological viewpoint in the book by Geiges [Gei06].

The easiest way to ensure that the hyperplane bundle is of the required type is to define it via a one-form that satisfies a certain equation. While we will always use the viewpoint that a contact structure is given by a one-form, let it be noted that there are other ways to define contact structures that do not single out a form.

Definition. A *contact structure* on an odd-dimensional smooth manifold M^{2m+1} is a one-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (d\eta)^m$ is nowhere vanishing (and thus a volume element), where $(d\eta)^m$ means m times the wedge product of $d\eta$ with itself. The form η is then called a *contact form* and (M, η) a contact manifold.

The contact form η defines a hyperplane bundle, the *contact distribution* $H \subset TM$, via $H_p = \ker \eta_p$. By the Frobenius theorem, this distribution would be integrable (or, equivalently, involutive), if $\eta \wedge d\eta = 0$. Thus, contact distributions are, as said before, "as far from being integrable as possible". As $\eta \wedge (d\eta)^m$ is nowhere vanishing, any $v \in H_p \setminus \{0\}$ satisfies $v \lrcorner d\eta_p \neq 0$. A contact manifold is always orientable because $\eta \wedge (d\eta)^m$ is a volume form. Thus, the normal bundle TM/H

of the contact distribution can be trivialised by a vector field. We fix one such field: The *Reeb vector field*, sometimes also called the characteristic vector field of the contact structure, is the unique vector field $\xi \in \mathfrak{X}(M)$ satisfying $\eta(\xi) = 1$ and $\xi \lrcorner d\eta = 0$.

We discuss some examples of contact structures, all taken from the book by Blair.

Example 1.1.1. The easiest example is the space $\mathbb{R}^{2m+1}$, with coordinates $(x, y, z) = (x_1, y_1, \dots, x_m, y_m, z)$ and the one-form

$$\eta = dz - \sum_{j=1}^{m} x_j dy_j.$$

One easily checks that this form is indeed a contact form on $\mathbb{R}^{2m+1}$. We have

$$H_{(x,y,z)} = \left\{ (u, v, w) \in \mathbb{R}^{2m+1} \;\middle|\; w = \sum_{j=1}^{m} x_j v_j \right\}$$

and $\xi = \partial_z$.

The space $\mathbb{R}^{2m+1}$ can also be considered as the *Heisenberg group* $\mathcal{H}^m$, where two isomorphic group structures are given by

$$(x, y, z) \cdot (\hat{x}, \hat{y}, \hat{z}) = \left(x + \hat{x}, y + \hat{y}, z + \hat{z} + \sum_{j=1}^{m} x_j \hat{y}_j \right). \tag{1.1}$$

and

$$(x, y, z) \cdot (\hat{x}, \hat{y}, \hat{z}) = \left(x + \hat{x}, y + \hat{y}, z + \hat{z} + \sum_{j=1}^{m} y_j \hat{x}_j - x_j \hat{y}_j \right). \tag{1.2}$$

We will encounter both formulations in what follows and therefore discuss the contact structure in either case. The form η as above gives a contact structure that is left-invariant under the first group action. The contact distribution is spanned by the left-invariant vector fields

$$X_j = \partial_{x_j}, \qquad\qquad Y_j = \partial_{y_j} + x_j \partial_z \tag{1.3}$$

and the Reeb vector field is $\xi = \partial_z$.

We come back to the group structure (1.2). Here, the above contact form is not invariant, we do however have the following contact form which is invariant under left group action

$$\eta_1 = dz + \sum_{j=1}^{m} (x_j dy_j - y_j dx_j).$$

A left-invariant basis for $H_1 = \ker(\eta_1)$ is the given by the vector fields

$$X_j = \partial_{x_j} + y_j \partial_z, \quad Y_j = \partial_{y_j} - x_j \partial_z$$

and the Reeb vector field is given by $\xi = \partial_z$.

On $\mathbb{R}^3$, there is another contact form given by

$$\eta = \sin(y) dx + \cos(y) dz.$$

Again, it is straightforward to check that this is indeed a contact form. As the form is 2π-periodic, it descends to a contact form on the torus $T^3 = \mathbb{R}^3 / (2\pi\mathbb{Z})^3$.

Finally, consider the sphere $S^{2m+1} \subset \mathbb{R}^{2m+2}$. Equip the euclidean space with coordinates $(x_0, y_0, \ldots, x_m, y_m)$ and define

$$\eta = \sum_{j=0}^{m} x_j dy_j - y_j dx_j.$$

Then, the restriction of η to the sphere is a contact structure and the Reeb vector field is given by $\xi = (-y_0, x_0, \ldots, -y_m, x_m)$. $\diamondsuit$

Just like the sphere, a large number of hypersurfaces of even-dimensional euclidean space admit a contact structure.

Proposition 1.1.2 ([Bla02, Thm 3.6]). *Let $M^{2m+1} \subset \mathbb{R}^{2m+2}$ be a smooth hypersurface and assume that for each $p \in M$, $(p + T_pM) \cap \{0\} = \emptyset$. Then, M carries a contact structure.*

As it turns out, every contact manifold locally looks like the one from our first example.

Proposition 1.1.3 ([Bla02, Thm 3.1]). *Let (M^{2m+1}, η) be a contact manifold. Then, around each point $p \in M$, there exists a neighbourhood U together with a chart giving local coordinates $(x_1, y_1, \ldots, x_m, y_m, z)$ such that in these coordinates*

$$\eta|_U = dz - \sum_{j=1}^{m} x_j dy_j.$$

This means in particular that there will be no local invariants of contact manifolds. In order to do geometry (in the Riemannian sense) on a contact manifold, we will want to equip it with a metric that is, in some sense, compatible with the contact structure. There are a number of concepts relating contact structures and Riemannian structures, cf [Bla02, chapter 4]. Here, we will only discuss the strongest one, that of a metric contact manifold.

Definition. A *metric contact manifold* is a tuple (M, g, η, J) with g a Riemannian metric on M, $\eta \in \Omega^1(M)$ and $J \in \mathrm{End}(TM)$ such that

(i) $\|\eta_x\| = 1 \quad$ for any $x \in M$,

(ii) $d\eta(X, Y) = 2g(JX, Y) \quad$ for any $X, Y \in \mathfrak{X}(M)$ and

(iii) $J^2 = -Id + \eta \otimes \eta^\natural$

Remark (Conventions for differential forms)**.** We note that we use the following convention for wedge products: For $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, we set

$$(\alpha\wedge\beta)(X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}\,(\sigma)\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)})\beta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}),$$

and thus, for the exterior differential, we obtain for $\alpha \in \Omega^1(M)$

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

Note that this differs from the usage in [Bla02] and [DT06] who follow the convention used by Kobayashi-Nomizu, which explains the additional 2 in (ii) compared with Blair's definition. Also, note the different sign convention in (ii) compared to [Bla02]. The different conventions are source of some confusion, as some authors use Blair's definition while using our convention for differential forms. This does not change anything essential, and we will in fact prove some of our results for the following larger class of manifolds.

Definition. For $\alpha > 0$, an *α-metric contact manifold* is a tuple (M, g, η, J) with g a Riemannian metric on M, $\eta \in \Omega^1(M)$ and $J \in \mathrm{End}(TM)$ such that

(i) $\|\eta_x\| = 1 \quad$ for any $x \in M$,

(ii) $d\eta(X, Y) = 2\alpha g(JX, Y) \quad$ for any $X, Y \in \mathfrak{X}(M)$ and

(iii) $J^2 = -Id + \eta \otimes \eta^\natural$

Remark. We are mainly interested in the cases $\alpha = 1, \frac{1}{2}$ which are the two possible definition of metric contact manifold appearing in the literature. We will show that our results on adapted connections hold for the whole class of α-metric contact manifolds but this class is not of interest to us beyond that.

Example 1.1.4. We consider the contact structure on the Heisenberg group from Example 1.1.1 with group structure (1.2). We have

$$d\eta = ddz + \sum_{j=1}^{m} dx_j \wedge dy_j - dy_j \wedge dx_j = 2\sum_{j=1}^{m} dx_j \wedge dy_j.$$

Now, if we equip $\mathbb{R}^{2m+1}$ (the underlying topological manifold of the Heisenberg group and its tangent spaces) with the family of endomorphisms

$$J_{(x,y,z)} = \begin{pmatrix} 0 & -1 & & & & 0 \\ 1 & 0 & & & & \\ & & \ddots & & & \vdots \\ & & & 0 & -1 & \\ & & & 1 & 0 & \\ -x_1 & -y_1 & \dots & -x_m & -y_m & 0 \end{pmatrix},$$

we have $JX_j = Y_j$ and $JY_j = -X_j$ and $J\xi = 0$. In particular J satisfies condition (iii) of a metric contact manifold. Moreover, for

$$g = \eta \otimes \eta + \sum_{j=1}^{m} dx_j^2 + dy_j^2,$$

we obtain a metric contact manifold.

For the group structure (1.1), we slightly change η to $\eta = 2dz - 2\sum_{j=1}^{m} x_j dy_j$ for ease of calculation and obtain

$$d\eta = -2\sum_{j=1}^{m} dx_j \wedge dy_j.$$

In particular, we have

$$d\eta(X_j, Y_j) = -2, \quad d\eta(X_j, Y_k) = d\eta(X_j, X_k) = d\eta(Y_j, Y_k) = 0.$$

For the endomorphisms J given by

$$J_{(x,y,z)} = \begin{pmatrix} 0 & 1 & & & & 0 \\ -1 & 0 & & & & \\ & & \ddots & & & \vdots \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ -x_1 & 0 & \dots & -x_m & 0 & 0 \end{pmatrix}. \tag{1.4}$$

we have $JX_j = -Y_j$ and $JY_j = X_j$. Setting $g = \frac{1}{2}d\eta(\cdot, J\cdot) + \eta \otimes \eta$, we obtain a metric contact manifold and see that $(X_j, Y_j, \xi = \frac{1}{2}\partial_z)$ form an orthonormal basis.

The sphere S^{2m+1} with the contact structure from Example 1.1.1 can also be made metric contact: We induce a Riemannian metric from the standard scalar product of $\mathbb{R}^{2m+2}$ and the endomorphism J from the standard complex structure J_0 of the surrounding space $\mathbb{C}^{m+1} \simeq \mathbb{R}^{2m+2}$. ◊

The torus $T^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ can also be made metric contact. We choose $\eta = \frac{1}{2}(\sin y \cdot dx + \cos y \cdot dz)$,

$$X = \partial_y, \quad Y = \cos y \cdot \partial_x - \sin y \cdot \partial_z \quad \text{and} \quad \xi = 2(\sin y \cdot \partial_x + \cos y \cdot \partial_z)$$

and define J by $JX = Y$, $JY = -X$ and $J\xi = 0$. Then, $(T^3, \frac{1}{4}\langle\cdot,\cdot\rangle, \eta, J)$ is a metric contact manifold and $2X, 2Y$ form an adapted basis of H.

We collect some elementary properties of α-metric contact manifolds that are easily checked.

Lemma 1.1.5. *Let (M, g, η, J) be an α-metric contact manifold. Then,*

1. *(M, η) is a contact manifold, i.e. $\eta \wedge (d\eta)^m \neq 0$.*
2. *The Reeb vector field ξ is the metric dual of η.*
3. *The endomorphism J stabilises the contact distribution and $J\xi = 0$.*

Proof. Using a local adapted basis $(e_1, f_1, \ldots, e_m, f_m, \xi)$, i.e. an ON basis where (e_j, f_j) span H and $f_j = Je_j$, and its dual (e^j, f^j, η), one can write $d\eta = 2\alpha \sum e^j \wedge f^j$ and the first claim follows. The other statements follow from direct calculations. □

We collect some further elementary identities. Using the relationship between $d\eta$ and g and the formula for J^2, we have

$$\begin{aligned} g(X, Y) &= -g(X, J^2Y) + \eta(X)\eta(Y) \\ &= -\frac{1}{2\alpha} d\eta(JY, X) + \eta(X)\eta(Y) \\ &= \frac{1}{2\alpha} d\eta(X, JY) + \eta(X)\eta(Y), \end{aligned}$$

i.e. the metric is completely determined by η via

$$g(X, Y) = \tfrac{1}{2\alpha} d\eta(X, JY) + \eta(X)\eta(Y). \tag{1.5}$$

Using similar calculations, we have the following elementary identities:

$$g(JX, Y) = -g(X, JY) \qquad g(JX, JY) = g(X, Y) + \eta(X)\eta(Y) \tag{1.6}$$
$$d\eta(JX, Y) = -d\eta(X, JY) \qquad d\eta(JX, JY) = d\eta(X, Y). \tag{1.7}$$

For any $X, Y \in \Gamma(H)$, we have

$$d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = -\eta([X, Y]). \tag{1.8}$$

Replacing Y by ξ, the same equation holds because $\eta(\xi)$ is a constant. Note that this does not imply that the equation holds for all vector fields as $\eta([X, Y])$ is not tensorial in X, Y. Equation (1.8) for $Y = \xi$ further implies $\eta([\xi, X]) = 0$ for any $X \in \Gamma(H)$ and thus

$$[\xi, \Gamma(H)] \subset \Gamma(H). \tag{1.9}$$

Due to the almost-complex structure on H, metric contact manifolds share some properties of almost-Hermitian manifolds. Among other things, we can introduce a Nijenhuis tensor.

Definition. Let (M, g, η, J) be an α-metric contact manifold. The (2,1)-tensor defined via

$$N(X,Y) = \tfrac{1}{4}\left([JX, JY] + J^2[X,Y] - J([X, JY] + [JX, Y])\right)$$

is called the *contact Nijenhuis tensor.*

While the vanishing of the Nijenhuis tensor in almost-Hermitian geometry implies that the almost-complex structure is integrable, no such straightforward interpretation exists in contact geometry. For metric contact manifolds with (almost) vanishing Nijenhuis tensor, see the section on CR manifolds. We finish our introduction to metric contact manifolds with two technical results that we will use later on.

Proposition 1.1.6 ([Bla02, Lemmas 6.1 and 6.2])**.** *Let (M, g, η, J) be an α-metric contact manifold. The endomorphism J has the following properties:*

1. *The Levi-Civita covariant derivative of J is, for any $X, Y, Z \in \mathfrak{X}(M)$, given by the following formula:*

$$2g((\nabla^g_X J)Y, Z) = g(JX, 4N(Y,Z)) + d\eta(JY, X)\eta(Z) + d\eta(X, JZ)\eta(Y).$$

In particular, $\nabla^g_\xi J = 0$.

2. *The Lie derivative $\mathcal{J} = \mathcal{L}_\xi J$, where*

$$\mathcal{L}_\xi J(X) = \mathcal{L}_\xi(JX) - J(\mathcal{L}_\xi X) = [\xi, JX] - J[\xi, X]$$

is symmetric with respect to g, traceless and anticommutes with J. Furthermore, we have

$$\nabla^g_X \xi = -\frac{1}{2} J\mathcal{J}X + \alpha JX.$$

Proof. While the result is known, we reproduce the proof (as it was worked out as part of the author's Diplom thesis [Sta11, Lemmas 2.1.10 and 2.1.11]) here to convince the reader that it carries over to α-metric contact manifolds. If X or Y is replaced by JV in (1.6), then the η-terms vanish. Thus, using the Koszul formula for ∇^g and the relationship between g and $d\eta$, we obtain

$$\begin{aligned}
2g((\nabla^g_X J)Y, Z) &= 2g(\nabla^g_X (JY), Z) + 2g(\nabla^g_X Y, JZ) \\
&= X(g(JY, Z)) + JY(g(X, Z)) - Z(g(X, JY)) \\
&\quad + g([X, JY], Z) + g([Z, X], JY) - g([JY, Z], X) \\
&\quad + X(g(Y, JZ)) + Y(g(X, JZ)) - JZ(g(X, Y)) \\
&\quad + g([X, Y], JZ) + g([JZ, X], Y) - g([Y, JZ], X) \\
&= X(\tfrac{1}{2\alpha} d\eta(Y, Z)) + JY(\tfrac{1}{2\alpha} d\eta(X, JZ)) + JY(\eta(X)\eta(Z)) \\
&\quad - Z(\tfrac{1}{2\alpha} d\eta(Y, X)) + \tfrac{1}{2\alpha} d\eta([X, JY], JZ) + \eta([X, JY])\eta(Z) \\
&\quad + \tfrac{1}{2\alpha} d\eta(Y, [Z, X]) - \tfrac{1}{2\alpha} d\eta(X, J[JY, Z]) - \eta(X)\eta([JY, Z]) \\
&\quad + X(\tfrac{1}{2\alpha} d\eta(Z, Y)) + Y(\tfrac{1}{2\alpha} d\eta(Z, X)) - JZ(\tfrac{1}{2\alpha} d\eta(X, JY)) \\
&\quad - JZ(\eta(X)\eta(Y)) + \tfrac{1}{2\alpha} d\eta(Z, [X, Y]) + \tfrac{1}{2\alpha} d\eta([JZ, X], JY) \\
&\quad + \eta([JZ, X])\eta(Y) - \tfrac{1}{2\alpha} d\eta([Y, JZ], JX) - \eta([Y, JZ])\eta(X).
\end{aligned}$$

Now, using that

$$\begin{aligned}
0 = dd\eta(A, B, C) &= A(d\eta(B, C)) - B(d\eta(A, C)) + C(d\eta(A, B)) \\
&\quad - d\eta([A, B], C) + d\eta([A, C], B) - d\eta([B, C], A)
\end{aligned}$$

for any vector fields $A, B, C \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned}
2g((\nabla^g_X J)Y, Z) &= \tfrac{1}{2\alpha} d\eta([Y, Z], X) - \tfrac{1}{2\alpha} d\eta([JY, JZ], X) + JY(\eta(X)\eta(Z)) \\
&\quad - JZ(\eta(X)\eta(Y)) + \eta([X, JY])\eta(Z) - \tfrac{1}{2\alpha} d\eta(X, J[JY, Z]) \\
&\quad - \eta([JY, Z])\eta(X) + \eta([JZ, X])\eta(Y) - \tfrac{1}{2\alpha} d\eta([Y, JZ], JX) \\
&\quad - \eta([Y, JZ])\eta(X) \\
&= \tfrac{1}{2\alpha} d\eta([Y, Z], X) - \tfrac{1}{2\alpha} d\eta([JY, JZ], X) + JY(\eta(X))\eta(Z) \\
&\quad + \eta(X) JY(\eta(Z)) - JZ(\eta(X))\eta(Y) - \eta(X) JZ(\eta(Y)) \\
&\quad + \eta([X, JY])\eta(Z) - \tfrac{1}{2\alpha} d\eta(X, J[JY, Z]) - \eta([JY, Z])\eta(X) \\
&\quad + \eta([JZ, X])\eta(Y) - \tfrac{1}{2\alpha} d\eta([Y, JZ], JX) - \eta([Y, JZ])\eta(X).
\end{aligned}$$

Then, using that for any vector fields $A, B \in \mathfrak{X}(M)$

$$d\eta(A, JB) = A(\underbrace{\eta(JB)}_{=0}) - JB(\eta(A)) - \eta([A, JB]),$$

we obtain

$$\begin{aligned}
2g((\nabla^g_X J)Y,Z) &= \tfrac{1}{2\alpha}d\eta([Y,Z],X) - \tfrac{1}{2\alpha}d\eta([JY,JZ],X) + d\eta(JY,X)\eta(Z) \\
&\quad + d\eta(JY,Z)\eta(X) - JZ(\eta(X))\eta(Y) + d\eta(Y,JZ)\eta(X) \\
&\quad - \tfrac{1}{2\alpha}d\eta(X,J[JY,Z]) + \eta([JZ,X])\eta(Y) - \tfrac{1}{2\alpha}d\eta([Y,JZ],JX) \\
&= -g(JX,[Y,Z]) + g(JX,[JY,JZ]) + d\eta(JY,X)\eta(Z) \\
&\quad + d\eta(X,JZ)\eta(Y) + \eta(X)(d\eta(JY,Z) + d\eta(Y,JZ)) \\
&\quad - g(JX,J[JY,Z]) - g(JX,J[Y,JZ]) \\
&= -g(JX,[Y,Z]) + g(JX,[JY,JZ]) - g(JX,J[JY,Z]) \\
&\quad - g(JX,J[Y,JZ]) + d\eta(JY,X)\eta(Z) + d\eta(X,JZ)\eta(Y)
\end{aligned}$$

Then, because $J^2 = -\operatorname{Id} + \eta\otimes\xi$, we obtain

$$\begin{aligned}
2g((\nabla^g_X J)Y,Z) &= g(JX,J^2[Y,Z]) - \eta([Y,Z])\eta(JX) + g(JX,[JY,JZ]) \\
&\quad - g(JX,J[JY,Z]) - g(JX,J[Y,JZ]) + d\eta(JY,X)\eta(Z) \\
&\quad + d\eta(X,JZ)\eta(Y) \\
&= g(JX,4N(Y,Z)) + d\eta(JY,X)\eta(Z) + d\eta(X,JZ)\eta(Y),
\end{aligned}$$

which proves the claim (1).

For (2), we first prove an auxiliary result: $\nabla^g_\xi \xi = 0$. Note that

$$\mathcal{L}_\xi \eta = d(\eta(\xi)) + \xi \lrcorner d\eta = 0$$

and thus

$$\begin{aligned}
0 = \mathcal{L}_\xi\eta(X) &= \xi(\eta(X)) - \eta([\xi,X]) \\
&= g(\nabla^g_\xi\xi,X) + g(\xi,\nabla^g_\xi X) - \eta(\nabla^g_\xi X - \nabla^g_X\xi) \\
&= g(\nabla^g_\xi\xi,X) - g(\xi,\nabla^g_X\xi).
\end{aligned}$$

Noting that ξ is a vector field of constant length and thus $g(\xi,\nabla^g_X\xi) = 0$, this yields the claimed equation.

Furthermore, we have $\nabla^g_\xi J = 0$ and thus

$$\begin{aligned}
g((\mathcal{L}_\xi J)(X),Y) &= g(\nabla_\xi(JX) - \nabla^g_{JX}\xi - J(\nabla^g_\xi X) + J(\nabla^g_X\xi),Y) \\
&= g(\underbrace{(\nabla^g_\xi J)(X)}_{=0} - \nabla^g_{JX}\xi + J(\nabla^g_X\xi),Y).
\end{aligned}$$

If $X = \xi$, this is zero. The same holds for $Y = \xi$ because

$$g(-\nabla^g_{JX}\xi + J\nabla^g_X\xi,\xi) = -(JX)(\underbrace{g(\xi,\xi)}_{=const}) + \underbrace{g(\xi,\nabla^g_{JX}\xi)}_{=0} - g(\nabla^g_X\xi,\underbrace{JX}_{=0}) = 0.$$

Thus, we now consider $X, Y \in \xi^\perp$. Then, we have

$$\begin{aligned} g((\mathcal{L}_\xi J)(X), Y) &= g(-\nabla^g_{JX}(\xi), Y) - g(\nabla^g_X \xi, JY) \\ &= -(JX)(g(\xi, Y)) + g(\xi, \nabla^g_{JX} Y) - X(g(\xi, JY)) + g(\xi, \nabla^g_X JY) \\ &= \eta(\nabla^g_{JX} Y) + \eta(\nabla^g_X JY). \end{aligned}$$

By (1.8), we have

$$\begin{aligned} \eta([JX, Y] + [X, JY]) &= -d\eta(JX, Y) - d\eta(X, JY) \\ &= 0 \end{aligned}$$

and hence

$$\begin{aligned} g((\mathcal{L}_\xi J)(X), Y) &= \eta(\nabla^g_{JX} Y) + \eta(\nabla^g_X JY) \\ &= \eta(\nabla^g_Y JX) + \eta(\nabla^g_{JY} X). \end{aligned}$$

Arguing as above, the right hand side is equal to $g(X, (\mathcal{L}_\xi J)(Y))$, which proves symmetry.

Next, by (1), we have

$$\begin{aligned} 2g((\nabla^g_X J)(\xi), Z) &= g(JX, 4N(\xi, Z)) + d\eta(X, JZ) \\ &= g(JX, J^2[\xi, Z] - J[\xi, JZ]) + d\eta(X, JZ) \\ &= -g(JX, J(\mathcal{L}_\xi J)(Z)) + 2\alpha g(JX, JZ). \end{aligned}$$

Next, using (1.6), we compute

$$\begin{aligned} 2g((\nabla^g_X J)(\xi), Z) &= -g(X, (\mathcal{L}_\xi J)(Z)) + \eta(X)\eta((\mathcal{L}_\xi J)(Z)) + 2\alpha g(Z, X) \\ &\quad - \tfrac{1}{2\alpha}\eta(Z)\eta(X) \\ &= -g((\mathcal{L}_\xi J)(X), Z) + \tfrac{1}{2\alpha} g(Z, X) - \tfrac{1}{2\alpha} g(\eta(X)\xi, Z), \end{aligned}$$

where the last equation follows because the symmetry of $\mathcal{J}$ implies that

$$\eta(\mathcal{J}(Z)) = g(\mathcal{J}(\xi), Z) = 0.$$

Therefore, we obtain the following equivalent statements:

$$\begin{aligned} (\nabla^g_X J)(\xi) &= -\frac{1}{2}(\mathcal{J})(X) + \alpha X - \alpha\eta(X)\xi, \\ -J(\nabla^g_X \xi) &= -\alpha(\mathcal{J})(X) + \alpha X - \alpha\eta(X)\xi - \nabla^g_X(\underbrace{J\xi}_{=0}), \\ \nabla^g_X \xi &= -\frac{1}{2}J(\mathcal{J})(X) + \alpha JX + \underbrace{\eta(\nabla^g_X \xi)}_{=g(\xi, \nabla^g_X \xi)=0}\xi, \\ \nabla^g_X \xi &= -\frac{1}{2}J\mathcal{J}X + \alpha JX. \end{aligned} \tag{1.10}$$

Hence, we obtain

$$\begin{aligned}
\tfrac{1}{2\alpha}g(X,JY) &= d\eta(Y,X)\\
&= Y(\eta(X)) - X(\eta(Y)) - \eta([Y,X])\\
&= g(\nabla^g_Y X,\xi) + g(X,\nabla^g_Y\xi) - g(\nabla^g_X Y,\xi) - g(Y,\nabla^g_X\xi) - g(\nabla^g_Y X,\xi)\\
&\quad + g(\nabla^g_X Y,\xi)\\
&= g(X,\nabla^g_Y\xi) - g(Y,\nabla^g_X\xi)\\
&\overset{(1.10)}{=} \tfrac12 g(X,J\mathcal{J}Y + 2\alpha JY) - \tfrac12 g(Y,J\mathcal{J}X + 2\alpha JX)\\
&= \tfrac12\big(g(X,J\mathcal{J}Y) - g(Y,J\mathcal{J}X)\big) + \tfrac{1}{2\alpha}g(X,JY),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 &= (g(X,J\mathcal{J}Y) - g(Y,J\mathcal{J}X))\\
&= g(X,J\mathcal{J}Y) + g(JY,\mathcal{J}X) = g(X,J\mathcal{J}Y) + g(X,\mathcal{J}JY),
\end{aligned}$$

which implies $J\mathcal{J} = -\mathcal{J}J$.

Considering the trace of $\mathcal{J}$, we note that it follows from $J\mathcal{J} = -\mathcal{J}J$ that for any eigenvector X and eigenvalue λ of $\mathcal{J}$ that

$$\mathcal{J}(JX) = -J(\mathcal{J}X) = -\lambda JX, \tag{1.11}$$

i.e. $-\lambda$ is also an eigenvalue. As J is an isomorphism of H and all eigenspaces associated with nonzero eigenvalues of $\mathcal{J}$ must be in H (because $\mathcal{J}\xi = 0$), the corresponding eigenspaces are of equal dimension. This implies that $\operatorname{tr}\mathcal{J} = 0$. $\square$

Corollary 1.1.7 ([Bla02, Corollary 6.2])**.** *On an α-metric contact manifold, the Reeb vector field is divergence-free.*

Proof. Using that $\nabla^g_X\xi = -\frac12 J\mathcal{J}X + \alpha JX$, we have

$$\operatorname{div}^g(\xi) = \sum_{j=1}^{2m+1} g(\nabla^g_{b_j}\xi, b_j) = \sum_{j=1}^{2m+1} g(-\tfrac12 J\mathcal{J}b_j + \alpha Jb_j, b_j),$$

where (b_j) is a local ON basis of TM. Now, assume that (b_j) is an eigenvector basis for $\mathcal{J}$. Then, the above expression becomes zero as $g(Jb_j, b_j) = 0$. $\square$

Contact structures on S^1-bundles

A large class of examples of metric contact manifolds can be obtains as S^1-bundles over almost-complex manifolds. They will later provide examples of manifolds where the horizontal Dirac operator is fairly easy to study.

We begin by briefly introducing the language of Riemannian submersions, which we shall use in the sequel. Throughout this section, let $\pi\colon (M,g) \to (\bar{M},\bar{g})$ be a Riemannian submersion.

In this setting, the tangent space of M splits into the horizontal and the vertical tangent spaces:

$$\begin{aligned} V_xM &= \{X \in T_xM \mid d\pi(X) = 0\} && \textit{vertical tangent space} \\ H_xM &= V_xM^{\perp} \simeq T_{\pi(X)}\bar{M} && \textit{horizontal tangent space}. \end{aligned}$$

Then, for every $X \in \mathfrak{X}(\bar{M})$, there is a unique vector field $X^* \in \mathfrak{X}(M)$ satisfying $X^*(x) \in H_xM$ and $d\pi(X^*(x)) = X(\pi(x))$ for every $x \in M$. The vector field X^* is called the *horizontal lift* of X.

Locally, HM is spanned by the horizontal lifts of vector fields on $\bar{M}$. However, beware that the space of horizontal vector fields $\Gamma(HM)$ differs from the space of horizontal lifts and is not isomorphic to the space of vector fields on $\bar{M}$. Any vector field can be written as $X = vX + hX$ with vX and hX the vertical and horizontal parts respectively. To compare the Levi-Civita connection on the two manifolds we introduce the following two *fundamental tensors*:

$$\begin{aligned} F^1_XY &= v(\nabla^g_{hX}hY) + h(\nabla^g_{hX}vY), \\ F^2_XY &= h(\nabla^g_{vX}vY) + v(\nabla^g_{vX}hY). \end{aligned}$$

Note that these tensors are often denoted by A and T. However, we will use these symbols for the potential and torsion of a connection later on (cf section 1.4.2). We then have the following results:

Lemma 1.1.8 ([FIP04, Lemma 1.1 and Proposition 1.5])**.** *Let $\pi\colon (M,g) \to (\bar{M},\bar{g})$ be a Riemannian submersion and let $X, Y \in \mathfrak{X}(\bar{M})$. Then we have*

(1) *$h[X^*,Y^*]$ is the horizontal lift of $[X,Y]$ and $v[X^*,Y^*] = 2F^1_{X^*}Y^*$.*

(2) *$h(\nabla^g_{X^*}Y^*)$ is the horizontal lift of $\nabla^{\bar{g}}_XY$.*

(3) *For any vertical vector field $V \in \Gamma(VM)$, $[X^*,V]$ is vertical.*

We now consider Riemannian submersions where the base manifold has an almost-complex structure and the fibre has type S^1 and discuss how to induce contact structures on the total space.

Let $(\bar{M},\bar{g},\bar{J})$ be almost-Hermitian, i.e. $\bar{J}$ is an almost-complex structure on $T\bar{M}$ ($\bar{J}^2 = -Id$) and the metric is compatible with it: $\bar{g}(\bar{J}X,\bar{J}Y) = \bar{g}(X,Y)$. We now consider an S^1-principal bundle $\pi : M \to \bar{M}$. Then, a connection form on M is a right-invariant one-form $i\eta \in \Omega^1(M,\mathfrak{s}^1 \simeq i\mathbb{R})$. We fix such a connection form $i\eta$ and chose a fundamental vector field ξ such that $\eta(\xi) = 1$. Furthermore, we define a metric on M via

$$g = \pi^* \bar{g} + \eta \otimes \eta.$$

Then $\pi : M \to \bar{M}$ is a Riemannian submersion, its vertical tangent space is trivialised by ξ, which is the metric dual of η, and the horizontal tangent space is given by the kernel of η.

Furthermore, we define an endomorphism field on TM via

$$JX = (\bar{J} d\pi(X))^*,$$

where * denotes the horizontal lift. Then the manifold (M, g, J, η) has the following properties:

Proposition 1.1.9 ([FIP04, Theorem 4.5]). *Let $(\bar{M}, \bar{g}, \bar{J})$ be almost-Hermitian and (M, g, J, η) be constructed as above. Then, the total space (M, g, J, η) is metric contact if and only if $(\bar{M}, \bar{g}, \bar{J})$ is symplectic, i.e. $\omega = \bar{g}(\bar{J}\cdot, \cdot)$ is a symplectic form on $\bar{M}$, which means that it is a closed, nondegenerate two-form.*

1.2 CR manifolds

We next turn our attention to CR manifolds and their relationship with metric contact manifolds. Again, the treatment will be rather concise, establishing mainly the facts that we will need later on. More on CR manifolds can be found in the book by Dragomir and Tomassini [DT06]. The basic model for a CR manifold is a real hypersurface $M^{2m+1} \subset \mathbb{C}^{m+1}$. The tangent space of such a hypersurface is not stable under the complex structure J_0, but we can define the stable tangent space as

$$H_p = T_pM \cap J_0(T_pM).$$

This defines a (real) codimension-one subbundle of TM and the complex structure restricts to an almost-complex structure J on H that satisfies the two conditions

$$[X, JY] + [JX, Y] \in \Gamma(H), \tag{1.12}$$

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0. \tag{1.13}$$

The sphere S^3 discussed in the examples of metric contact manifolds is an example of such a hypersurfaces. A CR (for complex-real or Cauchy-Riemann) manifold is now any manifold whose tangent bundle carries a similar structure.

Definition. A *CR manifold* is a smooth manifold M^{2m+1} together with a subbundle $H \subset TM$ of codimension one and an almost-complex structure J on H such that the formal integrability conditions (1.12) and (1.13) are satisfied.

The class of real hypersurfaces contains the class of boundaries of complex domains $\Omega \subset \mathbb{C}^{m+1}$ which are another classic example of CR manifolds.

For an alternative definition of a CR manifold also found in the literature, consider the complexified tangent bundle

$$TM_{\mathbb{C}} = TM \otimes \mathbb{C} = \coprod_{p \in M} \{ X + iY \mid X, Y \in T_pM \} .$$

Now, on a CR manifold (M, H, J), we also have the complexification $H_{\mathbb{C}} \subset TM_{\mathbb{C}}$ of the bundle H and extend the endomorphism J $\mathbb{C}$-linearly. Then, like the complexified tangent space of an almost-Hermitian manifold, $H_{\mathbb{C}}$ splits into the $\pm i$-eigenspaces of J:

$$H_{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)},$$
$$H^{(0,1)} = \overline{H^{(1,0)}},$$
$$H^{(1,0)} = \{ Z \in H_{\mathbb{C}} \mid JZ = iZ \} \quad \text{and} \quad H^{(0,1)} = \{ Z \in H_{\mathbb{C}} \mid JZ = -iZ \} .$$

Then, the half-spaces are given by $H^{(1,0)} = \{ X - iJX \mid X \in H \}$ and $H^{(0,1)} = \{ X + iJX \mid X \in H \}$ and have complex rank m and $H^{(1,0)}$ is involutive, i.e.

$$[\Gamma(H^{(1,0)}), \Gamma(H^{(1,0)})] \subset \Gamma(H^{(1,0)}).$$

Now, one can alternatively define a CR manifold by requiring that its complexified tangent space admit subbundles $H^{(1,0)}$, $H^{(0,1)}$ with the above properties. Then, one can define the bundle H as

$$H = \mathrm{Re}\,(H^{(1,0)} \oplus H^{(0,1)})$$

and the almost-complex structure on H via

$$J(X + \overline{X}) = i(X - \overline{X}).$$

Then, the involutivity condition is equivalent to the formal integrability conditions (1.12) and (1.13).

Again, we will want to do geometry (in the Riemannian sense) on our manifold, so we need a metric. We will produce one via a contact form: Let (M, H, J) be an oriented CR manifold. Then, we can find a one-form $\eta \in \Omega^1(M)$ whose kernel is H. Given such a form, we define the *Levi form* L_η on H via

$$L_\eta(X, Y) = \tfrac{1}{2} d\eta(X, JY).$$

Definition. If the Levi form is nondegenerate, we call (M, H, J, η) a nondegenerate CR manifold. If L_η is positive-definite, (M, H, J, η) is called a *strictly pseudoconvex* CR manifold

The nondegeneracy of the Levi form is enough to ensure that the one-form η is actually a contact form. In what follows, we will concentrate on strictly pseudoconvex CR manifolds because they come with a Riemannian metric defined by η, the so-called *Webster metric*. In order to define the Webster metric, we set $\pi_H(X) = X - \eta(X)\xi$, where ξ is the Reeb vector field of η, and then

$$g_\eta = \pi_H^* L_\eta + \eta \otimes \eta.$$

Alternatively, one could extend L_η to TM (with the same definition as on H), where we extend J by $J\xi = 0$, and because $\xi \lrcorner d\eta = 0$, we can then write $g_\eta = L_\eta + \eta \otimes \eta$.

Either way, g_η and the extended endomorphism J together with η give us a metric contact structure on the CR manifold.

The choice of the contact form η is not unique. Choosing $\tilde{\eta} = f\eta$ for some $f \in C^\infty(M, \mathbb{R}^+)$, we obtain

$$d\tilde{\eta} = df \wedge \eta + f d\eta$$

and thus, on H, $L_{\tilde{\eta}} = f L_\eta$ and the structure is strictly pseudoconvex again. We will discuss such changes at the end of this section.

Conversely, not any metric contact manifold comes from a CR manifold, because the conditions (1.12) and (1.13) are not necessarily satisfied. In fact, the metric contact structure comes from a CR structure (we will also sometimes say that the metric contact manifold *is* CR) if and only if the Nijenhuis tensor satisfies

$$JN(X,Y) = 0 \quad \text{for all } X, Y \in \Gamma(H). \tag{1.14}$$

So while we can consider strictly pseudoconvex CR manifolds as a subclass of metric contact manifolds, we can further restrict the class to obtain the Sasaki manifolds.

Definition. A *Sasaki manifold* is a Riemannian manifold (M^{2m+1}, g) together with a Killing vector field ξ satisfying the following conditions.

(i) $g(\xi, \xi) = 1$,

(ii) The endomorphism $J = -\nabla^g \xi$ satisfies $J^2 X = -X + g(X,\xi)\xi$ for any $X \in \mathfrak{X}(M)$.

(iii) $(\nabla^g_X J)Y = g(X,Y)\xi - g(Y,\xi)X$ for any $X, Y \in \mathfrak{X}(M)$.

Given a Sasaki manifold, a strictly pseudoconvex CR structure on the manifold is given by $\eta = g(\cdot, \xi)$, $H = \ker \eta$ and restricting J to H. Conversely, we have the following result that allows us to decide whether a CR manifold is Sasaki.

Proposition 1.2.1. *A strictly pseudoconvex CR manifold is Sasaki if and only if the Reeb vector field ξ is Killing or, equivalently, if and only if the Levi-Civita connection satisfies*

$$(\nabla^g_X J)Y = g(X,Y)\xi - \eta(Y)\xi \tag{1.15}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. By [Bla02, Thm 6.1, Thm 6.3], for a metric contact manifold we have an equivalence between

$$4N(X,Y) + d\eta(X,Y)\xi = 0 \tag{1.16}$$

and (1.15).

For $X, Y \in \Gamma(H)$, the first equation is equivalent to the CR integrability condition (1.13). For $X \in \Gamma(H)$ and $Y = \xi$ it is equivalent to

$$J(J[X,\xi] - [JX,\xi]) = J\mathcal{J}X = 0.$$

By [Bla02, Thm 6.2] this happen if and only if ξ is Killing. Thus, (1.16) is equivalent to the manifold being CR and ξ Killing. On the other hand, (1.15) is condition (iii) of the definition of a Sasaki manifold and the vanishing of $\mathcal{J}$ implies that $J = -\nabla^g\xi$ from the second statement of Proposition 1.1.6. □

We close this section by providing some examples of Sasaki and non-Sasaki CR manifolds.

Example 1.2.2. The sphere S^{2m+1} with the metric contact structure from example 1.1.4 is Sasaki. In fact, it is CR because it is a real hypersurface of complex space. Furthermore, ξ is Killing if and only if

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 0 \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

As S^{2m+1} is a hypersurface with induced Riemannian structure, we have $\nabla^g_X\xi = \mathrm{proj}_{TS}(X(\xi))$ and taking the scalar product with a vector field on S^{2m+1}, we can omit the projection to TS, i.e. we have the condition

$$\langle X(\xi), Y\rangle = -\langle X, Y(\xi)\rangle.$$

As $\xi = (-x_2, x_1, \ldots, -x_{2m+2}, x_{2m+1})$ can be extended to $\mathbb{R}^{2m+2}$, we can calculate its derivative in the usual way as

$$d\xi = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix},$$

which is skew-symmetric.

Consider the torus $T^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ with H defined as the kernel of $\eta = \frac{1}{2}(\sin y \cdot dx + \cos y \cdot dz)$ and J defined by $JX = Y$ and $JY = -X$ for $X = \partial_y$, $Y = \cos y \cdot \partial_x - \sin y \cdot \partial_z$ as in Example 1.1.4. Then,

$$[JX,Y] + [X,JY] = [Y,Y] - [X,X] = 0$$

and thus (1.12) is satisfied because it is skew-symmetric in X and Y and multiplying X or Y with a function only yields elements of H by the product rule for the vector field commutator. Condition (1.13) is also skew-symmetric and moreover tensorial in both arguments, so it suffices to check

$$[JX, JX] - [X, Y] - J([JX, Y] + [X, JY]) = -[Y, X] - [X, Y] = 0.$$

Moreover, the form η defines a strictly pseudoconvex structure, compare Example 1.1.4. The resulting strictly pseudoconvex CR structure is *not Sasaki*: The Sasaki property is equivalent to ξ being Killing. This is equivalent to

$$g(\nabla^g_U \xi, V) + g(U, \nabla^g_V \xi) = 0 \quad \text{for all } U, V \in \mathfrak{X}(T^3).$$

We calculate

$$\begin{aligned} X(\xi) &= 2(\cos y \cdot \partial_x - \sin y \cdot \partial_z) = 2Y, \\ Y(\xi) &= 0. \end{aligned}$$

Thus, we obtain

$$g(\nabla^g_U \xi, V) + g(U, \nabla^g_V \xi) = g(2Y, Y) + g(X, 0) = \frac{1}{2}$$

and therefore, ξ is not Killing.

The fact that metric contact structure on the torus above comes from a CR structure could also be deduced from the following more general result.

Lemma 1.2.3 ([Bla02, Corollary 6.4])**.** *Let (M, g, η, J) be a metric contact manifold of dimension three. Then, for $H = \ker \eta$, $(M, H, J|_H, \eta)$ is a strictly pseudoconvex CR manifold.*

Like for metric contact manifolds, S^1-bundles over complex manifolds provide a class of examples.

Example 1.2.4 (S^1-bundles)**.** We consider the metric contact structure on S^1-bundles from the previous section, i.e. we have an S^1-principal bundle $\pi\colon M \to \bar{M}$, where $(\bar{M}, \bar{g})$ is almost-Hermitian and the submersion $\pi\colon (M, g) \to (\bar{M}, \bar{g})$ is Riemannian. We fix a connection form $i\eta$ and lift the almost-complex structure $\bar{J}$ to an almost-complex structure on $H = \ker \eta$.

Proposition 1.2.5. *Let $(\bar{M}, \bar{g}, \bar{J})$ be almost-Hermitian and (M, H, J, η) be constructed as above. Then, (M, H, J, η) is a strictly pseudoconvex CR manifold if and only if $(\bar{M}, \bar{g}, \bar{J})$ is Kähler. In this case, the CR structure is Sasaki.*

Proof. In [Bla02, section 6.7.2], we have the following formula

$$[JX^*, JY^*] - [X^*, Y^*] - J[JX^*, Y] - J[X^*, JY^*] = \\ ([\bar{J}X, \bar{J}Y] - [X, Y] - \bar{J}[\bar{J}X, Y] - \bar{J}[X, \bar{J}Y])^*.$$

As the horizontal lifts form a basis of H and condition (1.13) is tensorial in both arguments, this implies that the CR condition (1.13) is equivalent to the integrability of the complex structure on $\bar{M}$. Using that $\bar{M}$ is symplectic if and only if M is metric contact, we have the equivalence between CR and Kähler. Furthermore, on any such structure ξ is Killing (again, see [Bla02, 6.7.2]) and thus the strictly pseudoconvex CR structure is actually Sasaki. □

Conformal change of the contact form

As we noted above, given a CR manifold (H, J), the choice of a strictly pseudoconvex structure is not unique. Given one strictly pseudoconvex structure η, setting $\tilde{\eta} = e^{2u}\eta$ gives another one.

Definition. Let (M, H, J, η), $(M, H, J, \tilde{\eta})$ be strictly pseudoconvex CR manifolds such that $\tilde{\eta} = e^{2u}\eta$ for some $u \in C^\infty(M)$. Then, we call the two structures *CR-conformal*.

As

$$d(\tilde{\eta}) = e^{2u}d\eta + 2e^{2u}du \wedge \eta,$$

the change of the Levi form (and thus the metric restricted to H) is conformal, whereas the change of the whole Webster metric g_η is not. In fact, if we define $g_\eta = \pi_H^* L_\eta + \eta \otimes \eta$, we must also change π_H. Alternatively, if we extend L_η and J and then simply write $g_\eta = L_\eta + \eta \otimes \eta$, we must note that J changes as well and the change of L_η is not conformal beyond H. To describe the changes to $\pi_H = Id - \eta \otimes \xi$ and J (which is extended via $J\xi = 0$), we must describe the Reeb vector field $\tilde{\xi}$ of the new contact form $\tilde{\eta}$. Given an orthonormal adapted frame (X_j, Y_j) of H for g_η, we obtain (cf also the formula in the proof of [DT06, Lemma 2.6])

$$\tilde{\xi} = e^{-2u}\left(\xi + \sum_{j=1}^{m} Y_j(u)X_j - X_j(u)Y_j\right). \tag{1.17}$$

Obviously, $\tilde{\eta}(\tilde{\xi}) = 1$. Furthermore,

$$\begin{aligned}\tilde{\xi} \lrcorner d\tilde{\eta} &= 2(du(\xi)\eta - du) + \sum_{j=1}^{m} Y_j(u) X_j \lrcorner d\eta - X_j(u) Y_j \lrcorner d\eta \\ &\qquad\qquad + 2\left(Y_j(u)du(X_j) - X_j(u)du(Y_j)\right)\eta \\ &= 2(du(\xi)\eta - du) + \sum_{j=1}^{m} Y_j(u) X_j \lrcorner d\eta - X_j(u) Y_j \lrcorner d\eta\end{aligned}$$

Checking the resulting one-form on the basis (X_j, Y_j, ξ), we obtain

$$\begin{aligned}(\tilde{\xi} \lrcorner d\tilde{\eta})(X_k) &= -2du(X_k) - X_k(u) d\eta(Y_k, X_k) = 0, \\ (\tilde{\xi} \lrcorner d\tilde{\eta})(Y_k) &= -2du(Y_k) + Y_k(u) d\eta(X_k, Y_k) = 0, \\ (\tilde{\xi} \lrcorner d\tilde{\eta})(\xi) &= -2(\xi(u) - du(\xi)) = 0,\end{aligned}$$

i.e. $\tilde{\xi} \lrcorner d\tilde{\eta} = 0$.

1.3 Differential forms on metric contact manifolds

This is a technical section that provides some results on the spaces of differential forms over a metric contact manifolds that we will use in the following section to describe the torsion tensor of an adapted connection. This section is taken in its entirety from the author's published paper [Sta12] with only minor changes to allow for α-metric contact manifolds.

To begin with, we note that, like the splitting of the tangent bundle on an almost-Hermitian manifold, the splitting on the complexified contact bundle

$$H_{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)}$$

induces a splitting on the complex-valued differential forms:

$$\Omega^k_{\mathbb{C}}(H) = \bigoplus_{p+q=k} \Omega^{p,q}(H),$$

where

$$\Omega^k_{\mathbb{C}}(H) = \Gamma(\Lambda^k H^*_{\mathbb{C}})$$

are complex-valued forms and

$$\Omega^{p,q}(H) = \Gamma(\Lambda^{p,q}(H^*)) = \Gamma(\Lambda^p(H^{(1,0)})^* \wedge \Lambda^p(H^{(0,1)})^*).$$

In what follows, we will concentrate on the spaces $\Omega^2(M, TM)$ of two-forms with values in TM and $\Omega^3(M)$ of real-valued three-forms. Before we begin the actual study of these spaces, we quickly introduce some conventions and operators that will be used in the following: For a TM-valued two-form B, we agree to write

$$B(X;Y,Z) := g(X, B(Y,Z)) \quad \text{for any } X,Y,Z \in TM. \tag{1.18}$$

Conversely, we may understand a three-form ω as a TM-valued two-form via

$$\omega(X,Y,Z) = g(X, \omega(Y,Z)). \tag{1.19}$$

Furthermore, we introduce the following operators: The Bianchi operator

$$\mathfrak{b}\colon \Omega^2(M,TM) \to \Omega^3(M)$$

given by

$$\mathfrak{b}B(X,Y,Z) = \tfrac{1}{3}\left(B(X;Y,Z) + B(Y;Z,X) + B(Z;X,Y)\right),$$

the operator

$$\begin{aligned} \mathfrak{M}\colon \Omega^2(M,TM) &\longrightarrow \Omega^2(M,TM) \\ B &\longmapsto B(J\cdot, J\cdot) \end{aligned}$$

and the trace operator

$$\operatorname{tr}\colon \Omega^2(M,TM) \longrightarrow \Omega^1(M)$$

given, for an ON basis (b_j) of TM, by

$$\operatorname{tr} B(X) = \sum_{j=1}^{2m+1} B(b_j; b_j, X).$$

Finally, the subspaces we are about to introduce will always be denoted by sub- and superscript indices. If we apply the same indices to a form, we mean its part in the respective subspace.

We have now set notation and begin considering the tangent bundle. Denoting $\Xi = \mathbb{R}\xi$, we see that the tangent bundle splits as $TM = H \oplus \Xi$ and thus, we have some induced splittings on the spaces of exterior powers:

$$\begin{aligned} TM \otimes \Lambda^2(T^*M) &= H \otimes \Lambda^2(H^*) \;\oplus\; \xi \otimes \Lambda^2(H^*) \;\oplus\; TM \otimes \eta \wedge H^*, \\ \Lambda^3(T^*M) &= \Lambda^3(H^*) \oplus \eta \wedge \Lambda^2(H^*). \end{aligned}$$

The theory developed by Paul Gauduchon for the respective forms over an almost Hermitian manifold carries over almost word-for-word to the bundles $H \otimes \Lambda^2(H^*)$ and $\Lambda^3(H^*)$. We will review these results and translate them to our case in a first subsection, and deal with the remaining spaces in a second subsection.

1.3.1 The forms over the contact distribution

In this part, we collect the results on the spaces $\Omega^2(H,H)$ and $\Omega^3(H)$. All manipulations we are about to perform on these spaces are pointwise and we will therefore use the bundles and spaces of sections indiscriminately. The calculations on this bundle are nearly equivalent to those on the tangent bundle of an almost-Hermitian manifold and thus we simply "translate" the results of [Gau97] to our case, omitting all proofs as they may be found in the original paper. Alternatively, one finds a detailed exposition in the first chapter of [Sta11].

To begin with, we introduce the following subspaces:

$$\Omega^{1,1}(H,H) := \{B \in \Omega^2(H,H) \mid \mathfrak{M}B = B\},$$
$$\Omega^{2,0}(H,H) := \{B \in \Omega^2(H,H) \mid B(JX,Y) = JB(X,Y)\ \forall X,Y \in \Gamma(H)\} \text{ and}$$
$$\Omega^{0,2}(H,H) := \{B \in \Omega^2(H,H) \mid B(JX,Y) = -JB(X,Y)\ \forall X,Y \in \Gamma(H)\}$$

and thus obtain the decomposition

$$\Omega^2(H,H) = \Omega^{1,1}(H,H) \oplus \Omega^{2,0}(H,H) \oplus \Omega^{0,2}(H,H).$$

Given a form $B \in \Omega^2(M,TM)$, we denote its part in $\Omega^2(H,H)$ as

$$B_H = B^{1,1} + B^{2,0} + B^{0,2}.$$

We note that $\Omega^{2,0}(H,H) \oplus \Omega^{0,2}(H,H)$ forms the eigenspace of $\mathfrak{M}$ to the eigenvalue -1. The image of $\Omega^2(H,H)$ under $\mathfrak{b}$ lies in $\Omega^3(H)$ and we will now study that space. It can be embedded into the space of complex forms $\Omega^3_{\mathbb{C}}(H) \simeq \Omega^3(H) \otimes \mathbb{C}$ and thus, any $\omega \in \Omega^3(H)$ admits a splitting into (complex) forms of type (p,q). We define

$$\omega^+ := \omega^{2,1} + \omega^{1,2},$$
$$\omega^- := \omega^{3,0} + \omega^{0,3}.$$

The reason why we consider these forms is that, as opposed to the simple parts of type (p,q), they are again real forms (i.e. real-valued when evaluated on elements of H). We define the respective spaces as

$$\Omega^+(H) := \{\omega \in \Omega^3(H) \mid \omega = \omega^+\},$$
$$\Omega^-(H) := \{\omega \in \Omega^3(H) \mid \omega = \omega^-\}.$$

Moreover, we have

Lemma 1.3.1 ([Gau97, p.262])**.** *Let $\omega \in \Omega^3(H)$. We also consider ω as an element of $\Omega^2(H,H)$ via equation* (1.19) *and it thus admits a splitting as $\omega = \omega^{1,1}+\omega^{2,0}+\omega^{0,2}$. Then the following relations are satisfied:*

$$\omega^+ = \omega^{2,0} + \omega^{1,1}, \qquad \omega^{2,0} = \tfrac{1}{2}\left(\omega^+ - \mathfrak{M}\omega^+\right),$$
$$\omega^- = \omega^{0,2}, \qquad \omega^{1,1} = \tfrac{1}{2}\left(\omega^+ + \mathfrak{M}\omega^+\right).$$

Furthermore, for an element of any of the subspaces of $\Omega^2(H,H)$, we can determine the type of its image under the Bianchi operator as the following lemma states more precisely.

Lemma 1.3.2 ([Gau97, section 1.4])**.**

(1) *Let $B \in \Omega^{0,2}(H,H)$. Then $\mathfrak{b}B \in \Omega^-(H)$.*

(2) *For any $B \in \Omega^{2,0}(H,H)$, we have $\mathfrak{b}B \in \Omega^+(H)$. Moreover $\mathfrak{b}|_{\Omega^{2,0}} \colon \Omega^{2,0}(H,H) \to \Omega^+(H)$ is an isomorphism and its inverse is given by*

$$(\mathfrak{b}|_{\Omega^{2,0}})^{-1}\omega = \tfrac{3}{2}\left(\omega - \mathfrak{M}\omega\right). \tag{1.20}$$

(3) *Let $\Omega^{1,1}_s(H,H)$ be the subspace of $\Omega^{1,1}(H,H)$ of elements vanishing under $\mathfrak{b}$ and $\Omega^{1,1}_a(H,H)$ its orthogonal (with respect to the metric g extended to forms in the usual way) complement. Then, $\mathfrak{b}|_{\Omega^{1,1}_a} : \Omega^{1,1}_a(H,H) \to \Omega^+(H)$ is an isomorphism with its inverse given by*

$$(\mathfrak{b}|_{\Omega^{1,1}_a})^{-1}(\omega) = \tfrac{3}{4}\left(\omega + \mathfrak{M}\omega\right). \tag{1.21}$$

(4) *Combining the above results, we see that for any $B \in \Omega^2(H,H)$ we have $(\mathfrak{b}B)^- = \mathfrak{b}(B^{0,2})$ and $(\mathfrak{b}B)^+ = \mathfrak{b}(B^{1,1} + B^{2,0})$. Furthermore, we obtain an isomorphism $\phi\colon \Omega^{2,0}(H,H) \to \Omega^{1,1}_a(H,H)$ given by*

$$\phi(B) = \tfrac{3}{4}(\mathfrak{b}B + \mathfrak{M}\mathfrak{b}B) \quad \text{and} \quad \phi^{-1}(A) = \tfrac{3}{2}(\mathfrak{b}A - \mathfrak{M}\mathfrak{b}A)$$

Corollary 1.3.3. (1) *Let $\omega \in \Omega^+(H)$. Then,*

$$\mathfrak{b}\mathfrak{M}\omega = \frac{1}{3}\omega. \tag{1.22}$$

(2) *Let $\omega \in \Omega^+(H)$. Then, $\mathfrak{M}\omega$ is not skew-symmetric in all three arguments.*

Proof. For the first claim, write $\omega = \omega^{1,1} + \omega^{2,0}$ and use that $\mathfrak{M}\omega^{1,1} = \omega^{1,1}$ and $\mathfrak{M}\omega^{2,0} = -\omega^{2,0}$. The claim then follows from a straightforward calculation. The second claim is then deduced as follows: Assume $\mathfrak{M}\omega$ skew-symmetric. Then, by (1), $\omega = 3\mathfrak{b}\mathfrak{M}\omega = 3\mathfrak{M}\omega$. Then, as $\omega = \omega^{1,1} + \omega^{2,0}$, we deduce

$$\mathfrak{M}\omega = \mathfrak{M}\omega^{1,1} + \mathfrak{M}\omega^{2,0} = \omega^{1,1} - \omega^{2,0} = \tfrac{1}{3}(\omega^{1,1} + \omega^{2,0}),$$

which is absurd. $\square$

Remark. The case of a 3-manifold

In the case of a metric contact 3-manifold ($m = 1$), the space $\Omega^3(H)$ vanishes. Furthermore, using a local adapted basis (e_1, f_1) of H, the space of TM-valued two-forms is locally spanned by $e_1 \otimes e^1 \wedge f^1$ and $f_1 \otimes e^1 \wedge f^1$. These forms are of type (1,1), vanish under $\mathfrak{b}$ and have trace f^1 and $-e^1$ respectively.

We now have all the links between the various subspaces of $\Omega^2(H, H)$ and $\Omega^3(H)$ needed and conclude this part, turning next to the forms that do not take their arguments exclusively in H.

1.3.2 The other parts

What is left to consider now are the parts of $\Omega^2(M, TM)$ for which ξ may appear as an argument or a value. First, we consider the elements of $\Omega^2(H, \Xi)$: Any element of this space has the form $\xi \otimes \alpha$, where $\alpha \in \Omega^2(H)$. Therefore, its image under $\mathfrak{b}$ is obviously given by

$$\mathfrak{b}(\xi \otimes \alpha) = \frac{1}{3}\eta \wedge \alpha \in \eta \wedge \Omega^2(H) \tag{1.23}$$

We can decompose $\Omega^2(H)$ as

$$\begin{aligned} \Omega^2(H) &= \Omega^2_+(H) \oplus \Omega^2_-(H), \text{ where} \\ \Omega^2_\pm(H) &= \{\alpha \in \Omega^2(H) \mid \alpha(J\cdot, J\cdot) = \pm\alpha\}. \end{aligned}$$

These spaces are again the eigenspaces of the involution $\mathfrak{M}$ (defined on $\Omega^2(H)$ just as before) to the eigenvalues 1 and -1. This may be regarded as a decomposition of $\Omega^2(H, \Xi)$ and then, by (1.23), is stable under $\mathfrak{b}$.

Finally, there remains a last part to be considered, the forms in

$$TM \otimes \eta \wedge H^* = H \otimes \eta \wedge H^* \ \oplus \ \xi \otimes \eta \wedge H^*.$$

Any element of $H \otimes \eta \wedge H^*$ may be interpreted as $\eta \wedge \Phi$, where Φ is an endomorphism of H and we understand this wedge product to mean

$$\eta \wedge \Phi(X; Y, Z) = \eta(Y)g(X, \Phi(Z)) - \eta(Z)g(X, \Phi(Y))$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where we extend Φ by $\Phi\xi = 0$. Then, Φ may be further decomposed according to its behaviour with respect to g and J. We write

$$\begin{aligned} \mathrm{End}_\pm(H) &:= \{F : H \to H \mid g(X, FY) = \pm g(FX, Y)\}, \\ \mathrm{End}^J_\pm(H) &:= \{F \in \mathrm{End}_\pm(H) \mid FJ = JF\}. \end{aligned}$$

The behaviour of the symmetric and skew-symmetric parts under the Bianchi operator is described in the following lemma.

Lemma 1.3.4. *Let $\Phi \in \mathrm{End}_+(H)$ and $\Psi \in \mathrm{End}_-(H)$. Then, we have*

$$\mathfrak{b}(\eta \wedge \Phi) = 0 \quad \text{and} \quad \mathfrak{b}(\eta \wedge \Psi)(\xi, X, Y) = \tfrac{2}{3} g(Y, \Psi X).$$

Proof. For any $F \in \mathrm{End}(H)$, we have

$$\begin{aligned} 3\mathfrak{b}(\eta \wedge F)(\xi, X, Y) &= \eta \wedge F(\xi; X, Y) + \eta \wedge F(X; Y, \xi) + \eta \wedge F(Y; \xi, X) \\ &= -g(X, FY) + g(Y, FX). \end{aligned}$$

Then, using symmetry and skew-symmetry respectively yields the claim. □

Summing up the various decompositions we have introduced above, we have the following decomposition for any element $B \in \Omega^2(M, TM)$:

$$B = B^{2,0} + B^{1,1} + B^{0,2} + \xi \otimes B^2_+ + \xi \otimes B^2_- + \eta \wedge B^1_+ + \eta \wedge B^1_- + \xi \otimes \eta \wedge B^1_{\mathbb{R}}, \tag{1.24}$$

where $B^1_+ \in \Omega^1_+(H, H) \simeq \mathrm{End}_+(H)$ is a symmetric endomorphism, $B^1_- \in \Omega^1_-(H, H) \simeq \mathrm{End}_-(H)$ a skew-symmetric one and $B^1_{\mathbb{R}} \in \Omega^1(H)$. We will sometimes group these parts as follows:

$$\begin{aligned} B^2 &:= B^2_+ + B^2_-, \\ B^1 &:= B^1_+ + B^1_- + B^1_{\mathbb{R}}. \end{aligned}$$

1.3.3 Application: Kähler form and Nijenhuis tensor

In this section, we apply the decomposition into parts to two forms associated with a metric contact manifold. These results will be used in the following section to describe the torsion tensor of adapted connections.

We begin by defining the *Kähler form*

$$F(X, Y) = g(JX, Y) = \frac{1}{2\alpha} d\eta(X, Y).$$

We will consider its Levi-Civita covariant derivative $\nabla^g F$ as an element of $\Omega^2(M, TM)$ via the conventions $(\nabla^g F)(X; Y, Z) = (\nabla^g_X F)(Y, Z)$ and (1.18). Then, for this form and the Nijenhuis tensor, we have the following decomposition (see also [Gau97, Proposition 1] for the almost-Hermitian model and [Nic05, pp 366f] for the Nijenhuis tensor).

Proposition 1.3.5. *The Nijenhuis tensor of an α-metric contact manifold has the following properties:*

(N1) *We have $N = N^{0,2} - \frac{1}{4}\xi \otimes d\eta - \frac{1}{4}\eta \wedge (J\mathcal{J})$, where we recall that $\mathcal{J} = \mathcal{L}_\xi J$.*

(N2) *N is trace-free.*

(N3) *$N^{0,2}$ vanishes under $\mathfrak{b}$.*

Furthermore, for $\nabla^g F$, we have the following properties:

(F1) *The following parts of $\nabla^g F$ vanish:*

$$(\nabla^g F)^{1,1} \equiv 0, \quad (\nabla^g F)^{2,0} \equiv 0 \quad \textit{and} \quad (\nabla^g_\xi F) \equiv 0.$$

(F2) *$(\nabla^g F)^{0,2}$ and $N^{0,2}$ determine each other via*

$$(\nabla^g F)^{0,2}(X; Y, Z) = 2N^{0,2}(JX; Y, Z)$$

for any $X, Y, Z \in \Gamma(H)$.

(F3) *$(\nabla^g F)^1 \in \Omega^1(H, H)$ and it is given by*

$$(\nabla^g F)^1 X = 2N^1(JX) + \alpha JX, \ \textit{ or, alternatively, by}$$
$$g((\nabla^g F)^1 X, Y) = g(JY, 2N^1(X)) + \tfrac{1}{2} d\eta(X, JY).$$

(F4) *Altogether, $(\nabla^g F)$ has the following form:*

$$\nabla^g F = 2N^{0,2}(J\cdot; \cdot, \cdot) + \eta \wedge (\nabla^g F)^1.$$

Proof. **(N1) and (N2)** For $Y, Z \in \Gamma(H)$, we have that

$$\begin{aligned} 4N(JY, Z) &= [J^2Y, JZ] + J^2[JY, Z] - J([J^2Y, Z] + [JY, JZ]) \\ &= -J[JY, JZ] + J[Y, Z] - [Y, JZ] + J^2[JY, Z]. \end{aligned}$$

Now, for $X, Y, Z \in \Gamma(H)$, this implies $g(X, N(JY, Z)) = g(JX, N(Y, Z))$. This implies that the part in $\Omega^2(H, H)$ is of type $(0, 2)$. Furthermore, because $J(TM) \subset H$, we have for $Y, Z \in \Gamma(H)$ that

$$\begin{aligned} 4g(\xi, N(Y, Z)) = g(\xi, [JY, JZ]) &= \eta([JY, JZ]) \\ &= -d\eta(JY, JZ). \end{aligned}$$

The explicit form of N^1 is an easy calculation and that it is symmetric follows by the symmetry of $\mathcal{J}$ and the fact that $J \circ \mathcal{J} = -\mathcal{J} \circ J$ (cf Lemma 1.1.6). (N2) follows immediately.

(F1) Let $X, Y, Z \in \Gamma(H)$. Then, we have

$$\begin{aligned}(\nabla^g F)(X;Y,Z) &= (\nabla^g_X F)(Y,Z)\\ &= X(F(Y,Z)) - F(\nabla^g_X Y, Z) - F(Y, \nabla^g_X Z)\\ &= X(g(JY,Z)) + g(\nabla^g_X Y, JZ) - g(JY, \nabla^g_X Z)\\ &= -X(g(Y,JZ)) + X(g(Y,JZ)) - g(Y, \nabla^g_X JZ)\\ &\quad - X(g(JY,Z)) + g(\nabla^g_X JY, Z)\\ &= X(g(Y,JZ)) - g(Y, \nabla^g_X JZ) + g(J(\nabla^g_X JY), JZ)\\ &= -X(F(JY,JZ)) + F(\nabla^g_X JY, JZ) + F(JY, \nabla^g_X JZ)\\ &= -(\nabla^g F)(X; JY, JZ).\end{aligned}$$

Thus, $(\nabla^g F)^{1,1} = 0$. Concerning $(\nabla^g_\xi F)$, we use that $\nabla^g_\xi J = 0$ (Lemma 1.1.6) to obtain

$$\begin{aligned}\nabla^g_\xi F(X,Y) &= \xi(F(X,Y)) - F(\nabla^g_\xi X, Y) - F(X, \nabla^g_\xi Y)\\ &= g(\nabla^g_\xi (JX), Y) + g(JX, \nabla^g_\xi Y) - g(J\nabla^g_\xi X, Y) - g(JX, \nabla^g_\xi Y)\\ &= 0.\end{aligned}$$

Furthermore, by a well-known formula for the exterior derivative, we have for any $X, Y, Z \in \Gamma(H)$ that

$$\begin{aligned}0 = dF(X,Y,Z) &= (\nabla^g_X F)(Y,Z) - (\nabla^g_Y F)(X,Z) + (\nabla^g_Z F)(X,Y)\\ &= 3\mathfrak{b}(\nabla^g F)(X,Y,Z),\end{aligned}$$

i.e. $\mathfrak{b}(\nabla^g F) = 0$. Now, using (1.20), we deduce that

$$(\nabla^g F)^{2,0} = \tfrac{3}{2}\left((\mathfrak{b}(\nabla^g F))^+ + \mathfrak{M}(\mathfrak{b}(\nabla^g F))^+\right) = 0.$$

This concludes the proof of (F1).

(F2) and (N3) Explicitly writing out N and then using that ∇^g is torsion-free and metric, we obtain for any $X, Y, Z \in \Gamma(H)$:

$$\begin{aligned}&4(N(JX;Y,Z) + N(JY;X,Z) - N(JZ;X,Y)) =\\ &\quad 2(-g(JZ, \nabla^g_{JX} JY) + g(Z, \nabla^g_{JX} Y) + g(JZ, \nabla^g_X Y) + g(Z, \nabla^g_X JY)).\end{aligned} \tag{1.25}$$

Note that we could write $N^{0,2}$ instead of N here as all other parts vanish for arguments in H. On the other hand, consider $(\nabla^g F)$. We know that $(\nabla^g F)^{1,1}$ and $(\nabla^g F)^{2,0}$ vanish. Thus, by the properties of $(0,2)$-forms, we obtain

$$\begin{aligned}2(\nabla^g F)^{0,2}(X; JU, Z) &= (\nabla^g F)(X; JU, Z) + (\nabla^g F)(JX; U, Z)\\ &= -g(JZ, \nabla^g_{JX} JU) + g(Z, \nabla^g_{JX} U) + g(JZ, \nabla^g_X U)\\ &\quad + g(Z, \nabla^g_X JU).\end{aligned}$$

Substituting $Y = JU$ and comparing this with (1.25) yields

$$(\nabla^g F^{0,2})(X;Y,Z) = N^{0,2}(JX;Y,Z) + N^{0,2}(JY;X,Z) - N^{0,2}(JZ;X,Y). \tag{1.26}$$

Using this, we obtain that

$$\mathfrak{b}N^{0,2}(X,Y,Z) = -\mathfrak{b}(\nabla^g F)^{0,2}(JX;Y,Z) = 0,$$

which proves (N3). Finally, using (1.26) we conclude that

$$\begin{aligned}(\nabla^g F)^{0,2}(X;Y,Z) &= N^{0,2}(JX;Y,Z) + N^{0,2}(JY;X,Z) - N^{0,2}(JZ;X,Y)\\ &= -3(\mathfrak{b}N^{0,2})(JX;Y,Z) + 2N^{0,2}(JX,Y,Z).\end{aligned}$$

Using that $\mathfrak{b}N^{0,2}$ vanishes, this yields (F2).

(F3) We have for any $X \in \Gamma(H)$ and $Y \in \mathfrak{X}(M)$ that

$$\begin{aligned}\eta \wedge (\nabla^g F)^1(Y;\xi,X) &= (\nabla^g_Y F)(\xi,X)\\ &= Y(F(\xi,X)) - F(\nabla^g_Y \xi, X) - F(\xi, \nabla^g_Y X)\\ &= g(\nabla^g_Y \xi, JX).\end{aligned}$$

Then, using Lemma 1.1.6, we deduce

$$\begin{aligned}2g((\nabla^g F)^1 X, Y) &= -2g(J\nabla^g_Y \xi, X) = 2g((\nabla^g_Y J)\xi, X)\\ &= g(JY, 4N(\xi,X)) + d\eta(J\xi,Y)\eta(X) + d\eta(Y,JX)\eta(\xi)\\ &= g(JY, 4N^1(X)) + d\eta(Y,JX) \qquad (1.27)\\ &= -g(Y, 4JN^1(X)) + 2\alpha g(JY,JX)\\ &= g(Y, 4N^1(JX)) + 2\alpha g(Y,X). \qquad (1.28)\end{aligned}$$

Now, (1.27) proves the second identity in (F3) and the last of the above equations the first one.

□

1.4 Adapted connections

In this section, we discuss adapted connections on α-metric contact manifolds, i.e. connections that parallelise the metric contact structure. In particular, we will completely describe the space of such connections via their torsion and discuss some examples, with a focus on the Tanaka-Webster connection. This section is taken in its entirety, with minor changes, from the author's published paper [Sta12].

1.4.1 Definition and basic properties

We begin by introducing adapted connections and discussing some basic properties. A connection is called adapted if it parallelises the metric contact structure, more precisely:

Definition. Let (M, g, η, J) be an α-metric contact manifold. Then, a connection ∇ is called *adapted* if it is metric and satisfies

$$\nabla J = 0, \quad \nabla \eta = 0 \quad \text{and} \quad \nabla \xi = 0.$$

In fact, this definition is redundant, as the following lemma shows.

Lemma 1.4.1. *Let (M, g, η, J) be an α-metric contact manifold.*

(1) *Let ∇ be a metric connection such that $\nabla J = 0$. Then ∇ is adapted.*

(2) *Let ∇ be adapted. Then, for any $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(H)$, the vector field $\nabla_X Y$ is again in $\Gamma(H)$.*

(3) *Let ∇ be adapted. Then, $d\eta$ is parallel under ∇.*

Proof. For (1), we only need to show that $\nabla \xi = 0$. That $\nabla \eta = 0$ is then immediate. We know that $0 = (\nabla J)\xi = \nabla(J\xi) - J(\nabla \xi)$. Because $J\xi = 0$, this implies $J(\nabla_X \xi) = 0$, i.e. $\nabla_X \xi = \lambda \xi$, with $\lambda \in C^\infty(M)$ for any vector field X. However, because ξ has constant length, $g(\nabla_X \xi, \xi) = 0$ and thus $\lambda \equiv 0$.

For (2), we then obtain $g(\xi, \nabla_X Y) = X(g(\xi, Y)) - g(\nabla_X \xi, Y) = 0$.

Concerning (3), we calculate for any $X, Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned}
(\nabla_X d\eta)(Y, Z) &= X(d\eta(Y, Z)) - d\eta(\nabla_X Y, Z) - d\eta(Y, \nabla_X Z) \\
&= 2\alpha X(g(JY, Z)) - d\eta(\nabla_X Y, Z) - d\eta(Y, \nabla_X Z) \\
&= 2\alpha(g(\nabla_X(JY), Z) + g(JY, \nabla_X Z)) - d\eta(\nabla_X Y, Z) - d\eta(Y, \nabla_X Z) \\
&= 2\alpha(g(J\nabla_X Y, Z) + g(JY, \nabla_X Z)) - d\eta(\nabla_X Y, Z) - d\eta(Y, \nabla_X Z) \\
&= 0.
\end{aligned}$$

This yields the claim. $\square$

Before we begin describing the class of adapted connections, we review the following, very useful, technical result.

Lemma 1.4.2. *Let (M, g, η, J) be an α-metric contact manifold and ∇ an adapted connection on M. Then, for every point $p \in M$ there exists an open neighbourhood U and a local ON-basis $(b_1, \dots, b_{2m+1})$ of $TM|_U$ such that*

$$\nabla_{b_j} b_k(p) = 0$$

for all $j, k \in 1, \dots, 2m+1$. Moreover, the basis can be chosen adapted, i.e. $b_{2j} = Jb_{2j-1}$ for $j = 1, \dots, m$ and $b_{2m+1} = \xi$.

Definition. A local basis as described in this lemma will be called *p-synchronous.*

Proof. Let U be a normal neighbourhood of p (with respect to any metric connection, in particular we can use the one for ∇^g), i.e. for each point $q \in U$ there exsists a geodesic γ_q such that $\gamma_q(0) = p$ and $\gamma_q(1) = q$. Now, fix a basis $(b_1^p, \dots, b_{2m+1}^p)$ of T_pM and define $b_j(q)$ to be the parallel transport of b_j^p along γ_q. As parallel transport is an isometry, the resulting vector fields form an ON basis at each point. Moreover, as J and ξ parallel under ∇, if we choose $(b_1^p, \dots, b_{2m+1}^p)$ adapted, the resulting vector fields will form an adapted basis at each point. □

Little is known about these connections so far, the most well-known example is the Tanaka-Webster connection in the case where the metric contact structure is induced by a strictly pseudoconvex CR manifold. It is defined by demanding that it be metric and explicitly giving its torsion. A generalisation of this connection to arbitrary metric contact manifolds has been constructed by Tanno [Tan89], which is, however, in general not adapted. Nicolaescu [Nic05] has constructed a different generalisation, which is indeed adapted and another adapted connection which induces the same Dirac operator as ∇^g. We shall return to these connections later.

1.4.2 The torsion tensor of an adapted connection

To any metric connection, we can associate two tensors, the *torsion tensor*

$$T \in \Omega^2(M, TM) \quad \text{given by} \quad T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and the *potential*

$$A \in \Omega^1(M, End_-(TM)) \quad \text{given by} \quad A_X Y = \nabla_X Y - \nabla^g_X Y.$$

We can consider A as a TM-valued two form, via

$$A(X; Y, Z) = g(A_X Y, Z)$$

with the usual conventions. Then, torsion and potential are related via

$$\begin{aligned} T &= -A + 3\mathfrak{b}A, \\ A &= -T + \frac{3}{2}\mathfrak{b}T. \end{aligned} \tag{1.29}$$

Thus, any metric connection is completely determined by its torsion, i.e. any 2-form with values in TM is the torsion tensor of a metric connection. In order to obtain an adapted connection, we need to impose additional restrictions. To this end, we study the various parts of the torsion tensor in the following theorem (cf [Gau97, Proposition 2] for the Hermitian model).

Theorem 1.4.3. *Let (M, g, η, J) be an α-metric contact manifold and ∇ an adapted connection. Then, its torsion tensor T has the following properties:*

(1) *The $(0,2)$-part is given by $T^{0,2} = N^{0,2}$, i.e. in particular independent of ∇.*

(2) *The following relationships are satisfied for the parts of type $(2,0)$ and $(1,1)$:*

$$T^{2,0} - \phi^{-1}(T^{1,1}_a) = 0, \text{ or, equivalently,}$$
$$\mathfrak{b}(T^{2,0} - T^{1,1}_a) = 0,$$

where ϕ is the isomorphism from Lemma 1.3.2, statement (3).

(3) *The part in $\Omega^2(H, \Xi)$ is independent of ∇ and given by*

$$T^2 = T^2_+ = d\eta.$$

(4) *We have the following results on the endomorphism T^1. Its symmetric part T^1_+ is independent of ∇ and given by*

$$T^1_+ = -\tfrac{1}{2} J\mathcal{J},$$

where we recall $\mathcal{J} = \mathcal{L}_\xi J$, while the skew-symmetric part T^1_- lies in $\mathrm{End}^J_-(H)$.

(5) *The part $T^1_{\mathbb{R}}$ vanishes.*

Conversely, for any $\omega \in \Omega^+(H)$, $B \in \Omega^{1,1}_s(H, H)$ and $\Phi \in \mathrm{End}^J_-(H)$, there exists an adapted connection, whose torsion tensor satisfies

$$(\mathfrak{b}T)^+ = \omega, \quad T^{1,1}_s = B \quad \text{and } T^1_- = \Phi.$$

The total torsion tensor then has the following form:

$$T = N^{0,2} + \tfrac{9}{8}\omega - \tfrac{3}{8}\mathfrak{M}\omega + B + \xi \otimes d\eta - \tfrac{1}{2}\eta \wedge (J\mathcal{J}) + \eta \wedge \Phi. \tag{1.30}$$

In the sequel, we will denote the adapted connection defined by ω, B and Φ as $\nabla(\omega, B, \Phi)$.

Proof. **First step:** We prove that ∇ is adapted if and only if it satisfies

$$A(X; Y, JZ) + A(X; JY, Z) = -(\nabla^g F)(X; Y, Z) \tag{1.31}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. To this end, we compute

$$\begin{aligned} A(X; Y, JZ) + A(X; JY, Z) &= g(\nabla_X Y - \nabla^g_X Y, JZ) + g(\nabla_X (JY) - \nabla^g_X (JY), Z) \\ &= g((\nabla_X J)Y, Z) - g(\nabla^g_X Y, JZ) - g(\nabla^g_X (JY), Z). \end{aligned}$$

On the other hand we have that

$$\begin{aligned}-(\nabla^g F)(X;Y,Z) &= -X(F(Y;Z)) + F(\nabla^g_X Y, Z) + F(Y, \nabla^g_X Z)\\ &= -g(\nabla^g_X(JY), Z) - g(JY, \nabla^g_X Z) - g(\nabla^g_X Y, JZ)\\ &\quad + g(JY, \nabla^g_X Z).\end{aligned}$$

This yields the claimed equivalence.

Second step: Using that $A = -T + \frac{3}{2}\mathfrak{b}T$, we deduce that (1.31) is equivalent to

$$T(X;Y,JZ) + T(X;JY,Z) - \tfrac{3}{2}(\mathfrak{b}T(X;Y,JZ) + \mathfrak{b}T(X;JY,Z)) = (\nabla^g F)(X;Y,Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Alternatively, using that

$$T(\xi;\xi,X) = g(\xi, \nabla_\xi X - \nabla_X \xi - [\xi, X]) = -\eta([\xi,X]) \overset{(1.9)}{=} 0,$$

which proves (5), this may be written as the system of equations

$$T(X;Y,JZ) + T(X;JY,Z) - \tfrac{3}{2}(\mathfrak{b}T(X;Y,JZ)+\mathfrak{b}T(X;JY,Z)) = (\nabla^g F)(X;Y,Z), \tag{1.32}$$

$$T(\xi;Y,JZ) + T(\xi;JY,Z) - \tfrac{3}{2}(\mathfrak{b}T(\xi;Y,JZ)+\mathfrak{b}T(\xi;JY,Z)) = 0, \tag{1.33}$$

$$T(X;\xi,JZ) - \tfrac{3}{2}\mathfrak{b}T(X;\xi,JZ) = (\nabla^g F)(X;\xi,Z) \tag{1.34}$$

for any $X, Y, Z \in \Gamma(H)$. Furthermore, using the results of section 1.3.1, we find that (1.32) is equivalent to the system

$$2T^{0,2}(JX;Y,Z) - 3(\mathfrak{b}T)^-(JX;Y,Z) = (\nabla^g F)^{0,2}(X;Y,Z), \tag{1.35}$$

$$-T^{2,0}(JX;Y,Z) - \tfrac{3}{2}((\mathfrak{b}T)^+(X;Y,JZ) + (\mathfrak{b}T)^+(X;JY,Z)) = 0. \tag{1.36}$$

Third step: We now prove the claims (1)-(4). To begin with, we obtain from (1.35) and Proposition 1.3.5 that

$$2T^{0,2}(JX;Y,Z) - 3(\mathfrak{b}T)^-(JX;Y,Z) = N^{0,2}(X;Y,Z). \tag{1.37}$$

Furthermore, we use a well-known formula for the exterior derivative and see that

$$\begin{aligned}0 = dF(X,Y,Z) &= X(g(JY,Z)) - Y(g(JX,Z)) + Z(g(JX,Y))\\ &\quad - g(J[X,Y],Z) + g(J[X,Z],Y) - g(J[Y,Z],X).\end{aligned}$$

Because ∇ is metric and by the definition of T, this can be seen to be equal to

$$\begin{aligned}0 &= g(\nabla_X JY, Z) + g(JY, \nabla_X Z) - g(\nabla_Y JX, Z) - g(JX, \nabla_Y Z) + g(\nabla_Z JX, Y)\\ &\quad + g(JX, \nabla_Z Y) + g(\nabla_X Y, JZ) - g(\nabla_Y X, JZ) - T(JZ;X,Y) - g(\nabla_X Z, JY)\\ &\quad + g(\nabla_Z X, JY) + T(JY;X,Z) + g(\nabla_Y Z, JX) - g(\nabla_Z Y, JX) - T(JX;Y,Z)\\ &= g(\nabla_X JY, Z) - g(\nabla_Y JX, Z) + g(\nabla_Z JX, Y) + g(\nabla_X Y, JZ) - g(\nabla_Y X, JZ)\\ &\quad - T(JZ;X,Y) + g(\nabla_Z X, JY) + T(JY;X,Z) - T(JX;Y,Z).\end{aligned}$$

Using that $\nabla J = 0$, we then obtain

$$\begin{aligned}
0 &= g(J(\nabla_X Y), Z) - g(J(\nabla_Y X), Z) + g(J(\nabla_Z X), Y) - g(J(\nabla_X Y), Z) \\
&\quad + g(J(\nabla_Y X), Z) - T(JZ; X, Y) - g(J(\nabla_Z X), Y) + T(JY; X, Z) \\
&\quad - T(JX; Y, Z) \\
&= -T(JZ; X, Y) + T(JY; X, Z) - T(JX; Y, Z).
\end{aligned}$$

Taking the (0,2)-part on both sides, we see that

$$0 = -3\mathfrak{b}T^{0,2}(JX; Y, Z) = -3(\mathfrak{b}T)^-(JX; Y, Z).$$

Inserting this into (1.37) yields (1).

Next, we use that $(\mathfrak{b}T)^+ = \mathfrak{b}(T_a^{1,1} + T^{2,0})$ to deduce from (1.36) that

$$\begin{aligned}
T^{2,0}(X; Y, Z) =& \tfrac{3}{4}(\mathfrak{b}T_a^{1,1} - \mathfrak{M}(\mathfrak{b}T_a^{1,1}))(JX; JY, Z) \\
&+ \tfrac{3}{4}(\mathfrak{b}T^{2,0} - \mathfrak{M}(\mathfrak{b}T^{2,0}))(JX; JY, Z).
\end{aligned}$$

Using Lemma 1.3.2, we obtain that

$$T^{2,0}(X; Y, Z) = \frac{1}{2}\left(T^{2,0}(JX; JY, Z) + \phi^{-1}(T_a^{1,1})(JX; JY, Z)\right),$$

which yields the first equality of (2). The equivalent formulation is obtained simply by applying $\mathfrak{b}$.

(3) is deduced from Lemma 1.4.1 and (1.8) using the following simple calculation:

$$\begin{aligned}
T(\xi; X, Y) &= g(\xi, \nabla_X Y - \nabla_Y X - [X, Y]) \\
&= -g(\xi, [X, Y]) = -\eta([X, Y]) = d\eta(X, Y).
\end{aligned}$$

Using (3) and equation (1.34), we obtain the following equivalent equations:

$$\begin{aligned}
g((\nabla^g F)^1 Y, X) &= T(X; \xi, JY) - \tfrac{1}{2}(T(X; \xi, JY) \\
&\quad + T(\xi; JY, X) - T(JY; \xi, X)), \\
g((\nabla^g F)^1 Y, X) + \tfrac{1}{2} d\eta(JY, X) &= \tfrac{1}{2}\left(g(X, T_+^1(JY) + T_-^1(JY))\right. \\
&\quad \left. + g(JY, T_+^1(X) + T_-^1(X))\right), \\
g((\nabla^g F)^1 Y, X) + \tfrac{1}{2} d\eta(JY, X) &= g(X, T_+^1(JY)).
\end{aligned}$$

By (F3) of Proposition 1.3.5, we deduce

$$\begin{aligned}
g(X, T_+^1(Y)) &= -g((\nabla^g F)^1(JY), X) + \tfrac{1}{2} d\eta(Y, X) \\
&= -g(JX, 2N^1(JY)) - \tfrac{1}{2} d\eta(JY, JX) + \tfrac{1}{2} d\eta(Y, X) \\
&= g(X, 2JN^1(JY)) \\
&= g(X, -\tfrac{1}{2} J\mathcal{J} Y).
\end{aligned}$$

This yields the result on T^1_+ in (4). Concerning T^1_-, we use (3) and the fact that $d\eta(\cdot, J\cdot) = -d\eta(J\cdot, \cdot)$, to reduce (1.33) to

$$-\tfrac{3}{2}((\mathfrak{b}T)(\xi; Y, JZ) + (\mathfrak{b}T)(\xi; JY, Z)) = 0,$$

which, by definition of $\mathfrak{b}$ is equivalent to

$$\begin{aligned} &T(\xi; Y, JZ) + T(Y; JZ, \xi) + T(JZ; \xi, Y) \\ +&T(\xi; JY, Z) + T(JY; Z, \xi) + T(Z; \xi, JY) = 0. \end{aligned}$$

Once more making use of (3) and the above property of $d\eta$, we see that this is equivalent to

$$\begin{aligned} &-g(Y, T^1_+(JZ) + T^1_-(JZ)) + g(JZ, T^1_+(Y) + T^1_-(Y)) \\ &-g(Z, T^1_+(JY) + T^1_-(JY)) + g(JY, T^1_+(Z) + T^1_-(Z)) = 0. \end{aligned}$$

Using the symmetry and skew-symmetry of the respective parts, one finally obtains the equivalent condition

$$g(JZ, T^1_-Y) + g(Z, T^1_-(JY)) = 0,$$

which completes the proof of (4).

Fourth step: We now prove the last claim. By the above arguments and the fact that $\mathfrak{b}N^{0,2} = 0$, we see that (1.32) is fulfilled if we choose $T^{0,2}, T^{1,1}_a, T^{2,0}$ according to the conditions above, i.e all other parts of T_H may be chosen freely. Now, assuming $(\mathfrak{b}T)^+ = \omega$ and $T^{1,1}_s = B$, we see that $\omega = \mathfrak{b}(T^{1,1}_a + T^{2,0})$ and obtain

$$\begin{aligned} \mathfrak{b}(T^{2,0}) &= \tfrac{1}{2}(\mathfrak{b}(T^{1,1}_a + T^{2,0}) + \mathfrak{b}(T^{2,0} - T^{1,1}_a)) \\ &= \tfrac{1}{2}\omega, \\ \mathfrak{b}(T^{1,1}_a) &= \tfrac{1}{2}(\mathfrak{b}(T^{1,1}_a + T^{2,0}) - \mathfrak{b}(T^{2,0} - T^{1,1}_a)) \\ &= \tfrac{1}{2}\omega. \end{aligned}$$

Thus, by equations (1.20) and (1.21), we can deduce

$$\begin{aligned} T^{2,0} &= \tfrac{3}{2}(\mathfrak{b}T^{2,0} - \mathfrak{M}\mathfrak{b}T^{2,0}) = \tfrac{3}{4}(\omega - \mathfrak{M}\omega), \\ T^{1,1}_a &= \tfrac{3}{4}(\mathfrak{b}T^{1,1}_a - \mathfrak{M}\mathfrak{b}T^{1,1}_a) = \tfrac{3}{8}(\omega + \mathfrak{M}\omega). \end{aligned}$$

As we have seen above, equations (1.33) and (1.34) are satisfied if and only if we choose T^1_+ as described above and $T^1_- \in \mathrm{End}^J_-(H)$ and $T^2 = d\eta$. The explicit description of T is obtained by putting together all of the above data. □

One might now use this result to construct certain "canonical connections", by setting $T^{1,1}_s$, $(\mathfrak{b}T)^+$ and T^1_- equal to certain forms geometrically defined on a metric contact manifold.

Remark. Note that, unlike in the Hermitian case, the Levi-Civita connection is never adapted. If it were, than $T = 0$ would have to satisfy the properties of the above theorem. However, $\xi \otimes d\eta$ never vanishes (due to the contact condition $\eta \wedge (d\eta)^m \neq 0$).

Remark. The case of a 3-manifold

Using the results of remark 1.3.1, we see that in this case ω does not appear. Furthermore, any endomorphism of H commuting with J is locally given by its value on e_1 (freely choosable) as its value on f_1 is then determined by the commutativity rule.

1.4.3 The (generalised) Tanaka-Webster connection and CR connections

Assume that (M, H, J, η) is a strictly pseudoconvex CR manifold. On such manifolds, one has a canonical choice for the adapted connection, namely the *Tanaka-Webster connection* ∇^η. This connection is defined as the metric connection whose torsion is given by

$$T(X,Y) = d\eta(X,Y)\xi, \tag{1.38}$$

$$T(\xi, X) = -\frac{1}{2}([\xi, X] + J[\xi, JX]) = -\tfrac{1}{2}J\mathcal{J}X \tag{1.39}$$

for any $X, Y \in \Gamma(H)$. Using that $A = -T + \frac{3}{2}\mathfrak{b}T$, the explicit torsion above and the general properties of adapted connections, we obtain that

$$\nabla^\eta_X Y = \nabla^g_X Y - \eta(\nabla^g_X Y)\xi \tag{1.40}$$

$$\nabla^\eta \xi = 0 \tag{1.41}$$

$$\nabla^\eta_\xi X = \nabla^g_\xi X - \frac{1}{2}\sum_{j=1}^{2m} d\eta(X, b_j)b_j \tag{1.42}$$

for any $X, Y \in \Gamma(H)$ and an ON basis (b_j) of H.

The part of the torsion given by the second equation is called the *pseudo-Hermitian torsion* and denoted $\tau(X) = T(\xi, X)$.

Lemma 1.4.4. *The pseudo-Hermitian torsion is symmetric with respect to g and traceless.*

Proof. The symmetry stems from the facts that $\mathcal{J}$ is symmetric, J antisymmetric and the two operators anticommute. Concerning tracelessness, choose a basis (b_j) of H consisting of eigenvectors of $\mathcal{J}$. As we saw in the proof of Proposition 1.1.6, this basis can be chosen such that $b_{2j} = Jb_{2j-1}$ and $\mathcal{J}b_{2j} = \lambda b_{2j} = -\mathcal{J}b_{2j-1}$. As $\tau(\xi) = 0$, we then have

$$\operatorname{tr}\tau = -\frac{1}{2}\sum_{j=1}^{2m} g(J\mathcal{J}b_j, b_j) = -\frac{1}{2}\sum_{k=1}^{m} g(J\lambda b_{2k-1}, b_{2k-1}) - g(J\lambda J b_{2k-1}, Jb_{2k-1}).$$

As $J^2|_H = -1$, it follows that the trace is zero. □

Two different generalisations of the Tanaka-Webster connection to metric contact manifolds may be found in the literature. The older one, described by Tanno in [Tan89, Prop 3.1], is not adapted if the manifold is not CR. The generalisation constructed by Nicolaescu in [Nic05, section 3.2] is always adapted and is given via its torsion tensor by

$$T = N + \xi \otimes d\eta + \tfrac{1}{4}\eta \wedge d\eta + \tfrac{1}{4}\eta \wedge (J - J\mathcal{J}), \tag{1.43}$$

where the differences between the formula noted here and the one in [Nic05] are due to different conventions (namely for N and for the wedge product of one-forms with endomorphisms).

We will now describe the Tanaka-Webster connection in terms of the defining data according to Theorem 1.4.3. We begin by noting that as $JN = 0$, we have $T^{0,2} = N^{0,2} = 0$. Furthermore, $T(H,H) \subset \Xi$, and therefore, we have to choose $\omega = 0$, $B = 0$ such that $T^{1,1} = T^{2,0} = 0$. The part $T^2 = d\eta$ is determined independently of ∇ anyway. Finally, τ lies in $\Omega^1_+(H,H)$ and thus, $T^1_- = 0$. We summarise our findings on the Tanaka-Webster connection in the following lemma, in which we also characterise its generalisation to metric contact manifolds.

Lemma 1.4.5. *The Tanaka-Webster connection of a strictly pseudoconvex CR manifold is given by the following defining data:*

$$(\mathfrak{b}T)^+ = 0, \quad T^{1,1}_s = 0 \quad \text{and} \quad T^1_- = 0.$$

Using the same defining data on a general metric contact manifold, one obtains the generalised Tanaka-Webster connection constructed in [Nic05, section 3.2], see (1.43).

Proof. We have already established the first statement and what remains to prove is the second one. To this end, we consider the torsion of that connection, given by (1.43). Noting that $\eta \wedge J = -\eta \wedge d\eta + \xi \otimes d\eta$, we obtain

$$\begin{aligned} T &= N + \xi \otimes d\eta + \tfrac{1}{4}\eta \wedge d\eta - \tfrac{1}{4}\eta \wedge (J\mathcal{J}) - \tfrac{1}{4}(\eta \wedge d\eta - \xi \otimes d\eta) \\ &= N^{0,2} + \xi \otimes d\eta - \tfrac{1}{2}\eta \wedge (J\mathcal{J}). \end{aligned}$$

This is the torsion of an adapted connection where all freely choosable parts are equal to zero. □

Thus, the generalisation of the Tanaka-Webster connection constructed by Nicolaescu is a very natural one. Note that the only difference between the CR Tanaka-Webster connection and the generalised one is the part $T^{0,2} = N^{0,2}$, which vanishes if the manifold is CR. Note also that it is precisely this part in which this generalisation differs from the one constructed by Tanno and which ensures that Nicolaescu's connection is adapted.

In what follows, by the *generalised Tanaka-Webster connection*, we always mean the adapted connection $\nabla(0,0,0)$. We will denote this connection by ∇^η. As it coincides with the Tanaka-Webster connection where the latter is defined (i.e. on strictly pseudoconvex CR manifolds), this notation is not ambiguous.

Using the complex description of a CR structure, one has an involutive space $H^{1,0}$. One may now ask oneself whether there are adapted connections that are torsion-free on this space.

Definition ([Nic05, p. 369])**.** An adapted connection on a CR manifold is called a *CR connection* if its torsion (extended $\mathbb{C}$-linearly to $H_{\mathbb{C}}$) satisfies

$$T(H^{1,0}, H^{1,0}) = 0.$$

An easy calculation shows that ∇^η is of that type and thus, this class is nonempty. In fact, using Theorem 1.4.3, we may give a complete description of this class:

Lemma 1.4.6. *An adapted connection $\nabla(\omega, B, \Phi)$ is CR if and only if $\omega = 0$.*

Proof. The space $H^{1,0}$ is given by elements of type $X - iJX$, where $X \in H$. Thus, we obtain the condition

$$0 = T(X - iJX, Y - iJY) = T(X,Y) - T(JX,JY) - i(T(JX,Y) + T(X,JY)),$$

which, because $T(X,Y)$ is a real vector, is equivalent to

$$T(X,Y) = T(JX,JY) \quad \text{and} \quad T(JX,Y) = -T(X,JY).$$

We only need to satisfy the first condition as it implies the second one. This first condition implies that $T^{2,0}$ and $T^{0,2}$ as well as T^2_- must vanish. Both T^2_- and $T^{0,2}$ vanish anyway, so we obtain the condition $T^{2,0} = 0$. From the proof of Theorem 1.4.3 we know that $T^{2,0} = \frac{3}{4}(\omega - \mathfrak{M}\omega)$. This yields $\omega = \mathfrak{M}\omega$. However, this would mean that $\omega = \mathfrak{b}\omega = \mathfrak{b}\mathfrak{M}\omega = \frac{1}{3}\omega$, which is absurd and thus $\omega = 0$. $\square$

1.4.4 Adapted connections with skew-symmetric torsion

A lot of attention has recently been devoted to connections with skew-symmetric torsion. The existence of such connections adapted to almost metric contact manifolds has been discussed in [FI02, section 8]. This section may not add any new results, but we think a review of these results under a different light might still be interesting:

We use (1.30) and note that $\mathfrak{b}B = 0$ and $\mathfrak{b}N^{0,2} = 0$. Furthermore, because $J\mathcal{J}$ is symmetric, $\mathfrak{b}(\eta \wedge J\mathcal{J}) = 0$. Then, using (1.22), we deduce for the torsion T of $\nabla(\omega, B, \Phi)$ that

$$\mathfrak{b}T = \omega + \tfrac{1}{3}\eta \wedge d\eta + \mathfrak{b}(\eta \wedge \Phi). \tag{1.44}$$

Decomposing $\mathfrak{b}T$ into its various parts and using the results of section 1.3, we obtain that

$$(\mathfrak{b}T)^{0,2} = 0, \quad (\mathfrak{b}T)^{2,0} = \tfrac{1}{2}(\omega - \mathfrak{M}\omega), \quad (\mathfrak{b}T)^{1,1}_a = \tfrac{1}{2}(\omega + \mathfrak{M}\omega), \quad (\mathfrak{b}T)^{1,1}_s = 0,$$
$$(\mathfrak{b}T)^2_+ = \tfrac{1}{3}d\eta + \tfrac{2}{3}g(\Phi\cdot,\cdot), \quad (\mathfrak{b}T)^2_- = 0.$$

Comparing this with T yields the vanishing of $N^{0,2}$, $\mathfrak{M}\omega$ (and thus of ω), B and $\eta \wedge (J\mathcal{J})$ and implies that $\Phi = 2J$. One easily verifies that under these conditions, the remaining parts of T and $\mathfrak{b}T$ coincide as well. Recalling the structure of N and (1.14), we see that the manifold must be CR. If $\eta \wedge (J\mathcal{J})$ vanishes, so must $J\mathcal{J}$ and then, because J is an isomorphism on H, we have $\mathcal{J} = 0$. By (the proof of) Proposition 1.2.1 this is equivalent to the manifold being Sasaki. In conclusion, we have

Proposition 1.4.7. *A metric contact manifold (M, g, η, J) admits an adapted connection with skew-symmetric torsion if and only if it is Sasaki. In that case, this connection is unique and given by*

$$\omega = 0, \quad B = 0, \quad \Phi = 2J.$$

1.5 The curvature tensors of an adapted connection

In this section, we discuss the curvature tensors of adapted connections. We check which symmetry properties of the Riemannian curvature tensor are kept and which are lost and discuss the horizontal curvature tensors, i.e. curvature tensors that only act on vectors of H. Most of the material in this section is known and we only collect it here. We begin by recalling the standard definitions of curvature tensors.

Definition. Let ∇ be a metric connection on a Riemannian manifold (M, g). Then, we define the (3,1)-curvature tensor

$$\mathcal{R}^{\nabla}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and the (4,0)-curvature tensor

$$R^{\nabla}(X,Y,U,V) = g(\mathcal{R}^{\nabla}(X,Y)U, V).$$

Now, let (b_j) be an orthonormal basis of TM. Then, we can contract the curvature tensor to obtain the *Ricci tensor*

$$\mathrm{Ric}^{\nabla}(X,Y) = \sum_{j=1}^{n} R^{\nabla}(X, b_j, b_j, Y)$$

and the *scalar curvature*

$$\mathrm{scal}^{\nabla} = \sum_{j=1}^{n} \mathrm{Ric}(b_j, b_j).$$

Notation. We will write $\mathcal{R}^g$, R^g etc. for the curvature tensors of the Levi-Civita connection and $\mathcal{R}^\eta$, R^η etc. for the curvature tensors of the Tanaka-Webster connection of a strictly pseudoconvex CR manifold (M, H, J, η).

It is the easy to check the following well-known symmetry properties.

Lemma 1.5.1. *Let ∇ be a metric connection on a Riemannian manifold (M, g). Then, the curvature tensor R^∇ is skew-symmetric in the first two as well as in the last two arguments, i.e. we have*

$$R^{\nabla}(X,Y,U,V) = -R^{\nabla}(Y,X,U,V) = R^{\nabla}(Y,X,V,U).$$

Proof. The proof is the same as for the curvature tensor of the Levi-Civita connection. We reproduce it here to convince the reader that it holds for general metric connections. The skew-symmetry in the first two arguments is immediate from the definition. For the last two arguments, we have

$$\begin{aligned}
R^{\nabla}(X,Y,U,U) &= g(\nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U, U)\\
&= Xg(\nabla_Y U, U) - g(\nabla_Y U, \nabla_X U) - Y(g(\nabla_X U, U) + g(\nabla_X U, \nabla_Y U)\\
&\quad - [X,Y](g(U,U)) + g(U, \nabla_{[X,Y]}U)\\
&= X(Y(g(U,U)) - X(g(U,\nabla_Y U)) - Y(X(g(U,U)) + Y(g(U,\nabla_X U))\\
&\quad - [X,Y](g(U,U)) + g(U, \nabla_{[X,Y]}U)\\
&= -g(\nabla_X U, \nabla_Y U) - g(U, \nabla_X\nabla_Y U) + g(\nabla_Y U, \nabla_X U)\\
&\quad + g(Y, \nabla_Y\nabla_X U) + g(U, \nabla_{[X,Y]}U)\\
&= -R^{\nabla}(X,Y,U,U).
\end{aligned}$$

Thus, $R^\nabla(X,Y,U,U) = 0$ and this is equivalent to the required skew-symmetry. $\square$

The remaining symmetry properties are lost, in particular, the Bianchi identity does not hold anymore and has to be replaced by some formula involving the torsion tensor. We now focus on adapted connections.

Lemma 1.5.2. *Let (M^{2m+1}, H, J, η) be a metric contact manifold and ∇ an adapted connection. Then, we have*

$$\mathcal{R}^{\nabla}(X,Y)\xi = 0 \quad \text{and} \quad R^{\nabla}(X,Y,W,\xi) = 0.$$

for any $X, Y, W \in TM$.

In particular, in the definition of the Ricci and scalar curvature one may replace the orthonormal basis (b_j) by one of H.

$$\begin{aligned} \mathrm{Ric}^{\nabla}(X,Y) &= \sum_{j=1}^{2m} R^{\nabla}(X, b_j, b_j, Y) \\ \mathrm{scal}^{\nabla} &= \sum_{j=1}^{2m} \mathrm{Ric}(b_j, b_j) \end{aligned}$$

Proof. The first formula follows from the fact that $\nabla\xi = 0$. The second one follows from the first by skew-symmetry of R^{∇} in the last two arguments. The formulae for Ricci and scalar curvature are an immediate consequence. $\square$

If we restrict ourselves to the Tanaka-Webster connection, we can derive useful formulae comparing its curvature with the usual Riemannian curvature. See also [DT06, Thm 1.6] and [Bau99, Prop 16] for similar formulae.

Lemma 1.5.3. *Let (M, H, J, η) be a strictly pseudoconvex CR manifold and ∇^{η} its Tanaka-Webster adapted connection. Let $LX := \nabla^g_X \xi = \tau(X) + JX$ for $X \in TM$, where $\tau(X) = T(\xi, X)$ is the pseudo-Hermitian torsion. Then, for $Y, U, V, W \in \Gamma(H)$ we have*

$$\begin{aligned} R^{\eta}(Y,U,V,W) &= R^g(Y,U,V,W) + \tfrac{1}{2} d\eta(Y,U) d\eta(V,W) + g(g(LY,V)LU \\ &\quad - g(LU,V)LY, W), \\ R^{\eta}(Y,\xi,U,V) &= R^g(Y,\xi,U,V). \end{aligned}$$

Proof. The formula for L is already known, see (1.10) and the definition of the torsion of ∇^{η}.

We now turn to the comparison formulae for the curvature tensor. From the explicit description of ∇^{η} in (1.40) and the properties of an adapted connection, we see that

$$\begin{aligned} \nabla^{\eta}_Y \nabla^{\eta}_U V &= \nabla^g_Y(\nabla^{\eta}_U V) - \eta(\nabla^g_Y(\nabla^{\eta}_U V))\xi \\ &= \nabla^g_Y(\nabla^g_U V - \eta(\nabla^g_U V)\xi) - \eta(\nabla^g_Y[\nabla^g_U V - \eta(\nabla^g_U V)\xi])\xi \\ &= \nabla^g_Y \nabla^g_U - Y(\eta(\nabla^g_U V))\xi - \eta(\nabla^g_U V)LY - \eta(\nabla^g_Y[\nabla^g_U V - \eta(\nabla^g_U V)\xi])\xi. \end{aligned}$$

The ξ-part vanishes when taking the scalar product with $V \in H$. Furthermore,

$$\eta(\nabla^g_U V) = g(\nabla^g_U V, \xi) = Ug(V,\xi) - g(V, LU) = -g(V, LU)$$

On the other hand, by (1.42)

$$\begin{aligned}\nabla^\eta_{[Y,U]} V &= \nabla^\eta_{\pi_H([Y,U])} V + d\eta(Y,U)\nabla^\eta_\xi V \\ &= \nabla^g_{[Y,U]} V - \eta(\nabla^g_{[\pi_H([Y,U])} V)\xi - \frac{1}{2}\sum_{j=1}^{2m} d\eta(Y,U) d\eta(V, b_j) b_j.\end{aligned}$$

Taking the scalar product with W yields the first equation for ∇^η.

Next, using (1.42) and a p-synchronous frame (b_j), we obtain that at p,

$$\begin{aligned}\nabla^\eta_Y \nabla^\eta_\xi U &= \nabla^\eta_Y \left(\nabla^g_\xi U - \tfrac{1}{2}\sum_{j=1}^{2m} d\eta(U, b_j) b_j\right) \\ &= \nabla^g_Y \nabla^g_\xi U - \eta(\nabla^g_Y \nabla^g_\xi U)\xi - \tfrac{1}{2}\sum_{j=1}^{2m} Y(d\eta(U, b_j)) b_j \\ &= \nabla^g_Y \nabla^g_\xi U - \eta(\nabla^g_Y \nabla^g_\xi U)\xi - \tfrac{1}{2}\sum_{j=1}^{2m} d\eta(\nabla^\eta_Y U, b_j) b_j.\end{aligned}$$

We further have

$$\begin{aligned}\nabla^\eta_\xi \nabla^\eta_Y U &= \nabla^g_\xi(\nabla^\eta_Y U) - \tfrac{1}{2}\sum_{j=1}^{2m} d\eta(\nabla^\eta_Y U, b_j) b_j \\ &= \nabla^g_\xi \nabla^g_Y U - \nabla^g_\xi(\eta(\nabla^g_Y U)\xi) - \tfrac{1}{2}\sum_{j=1}^{2m} d\eta(\nabla^\eta_Y U, b_j) b_j \\ &= \nabla^g_\xi \nabla^g_Y U - \xi(\eta(\nabla^g_Y U))\xi - \tfrac{1}{2}\sum_{j=1}^{2m} d\eta(\nabla^\eta_Y U, b_j) b_j.\end{aligned}$$

Finally, $[\xi, Y]$ is in H and thus

$$\nabla^\eta_{[Y,\xi]} U = \nabla^g_{[Y,\xi]} U - \eta(\nabla^g_{[Y,\xi]} U)\xi.$$

Then, taking the scalar product with $V \in H$ yields the remaining equations. $\square$

From the comparison with the Riemannian tensor, we can deduce the following Bianchi-type identities for R^η. To simplify notation, we write $\mathfrak{b}_{Y,U,V}$ for the cyclic sum over Y, U and W, i.e.

$$\mathfrak{b}_{Y,U,V} A(Y,U,V,\cdot) = A(Y,U,V,\cdot) + A(U,V,Y,\cdot) + A(V,Y,U,\cdot)$$

for any tensor A with at least three arguments.

Lemma 1.5.4 ([Pet05, p. 231]). *Let (M, H, J, η) be a strictly pseudoconvex CR manifold and ∇^η its Tanaka-Webster connection. Then, the curvature tensor R^η satisfies the following Bianchi-type identities for any $U, V, W \in \Gamma(H)$:*

$$\begin{aligned} \mathfrak{b}_{U,V,W}\mathcal{R}^\eta(U,V)W &= \mathfrak{b}_{U,V,W}d\eta(U,V)\tau(W), \\ \mathfrak{b}_{\xi,U,V}\mathcal{R}^\eta(\xi,U)V &= \mathcal{R}^\eta(\xi,U)V + \mathcal{R}^\eta(V,\xi)U = (\nabla^g_V L)U - (\nabla^g_U L)V. \end{aligned} \tag{1.45}$$

Proof. In either formula, when taking a scalar product with ξ, both sides are zero. We begin (naturally enough) with the first formula. Considering the part in H, we obtain from the comparison with R^g that

$$\mathfrak{b}_{U,V,W}\mathcal{R}^\eta(U,V)W = \mathfrak{b}_{U,V,W}d\eta(U,V)JW + \mathfrak{b}_{U,V,W}(g(LU,W)LV - g(LV,W)LU).$$

As τ is symmetric and J is skew-symmetric, the last cyclic sum becomes

$$-2\mathfrak{b}_{U,V,W}g(U,JW)LV = -\mathfrak{b}_{U,V,W}d\eta(W,U)LV.$$

As $L = \tau + J$, this yields the first claim.

For the second claim, we again use the comparison with R^g and obtain from the Bianchi identity for that curvature tensor that

$$\mathfrak{b}_{\xi,U,V}\mathcal{R}^\eta(\xi,U)V = \mathcal{R}^\eta(\xi,U)V + \mathcal{R}^\eta(V,\xi)U = -\mathcal{R}^g(U,V)\xi.$$

This curvature tensor is obtained as

$$\begin{aligned} \mathcal{R}^g(U,V)\xi &= \nabla^g_U\nabla^g_V\xi - \nabla^g_V\nabla^g_U\xi - \nabla^g_{[U,V]}\xi \\ &= \nabla^g_U(LV) - \nabla^g_V(LU) - L[U,V] \\ &= (\nabla^g_U L)V - (\nabla^g_V L)U. \end{aligned}$$

This yields the claim. $\square$

Applying the Bianchi identity to $R^\eta(Y,U,V,W)$, $R^\eta(U,V,W,Y)$, $R^\eta(V,W,Y,U)$ and $R^\eta(W,Y,U,V)$ and and adding the equations, one deduces that for $Y, U, V, W \in \Gamma(H)$, one has (cf [Pet05, p. 231f.])

$$\begin{aligned} R^\eta(Y,U,V,W) = R^\eta(V,W,Y,U) &+ d\eta(U,V)g(\tau Y,W) + d\eta(Y,W)g(\tau U,V) \\ &+ d\eta(V,Y)g(\tau U,W) + d\eta(W,U)g(\tau Y,V). \end{aligned} \tag{1.46}$$

While this symmetry of the curvature tensor fails, we have for $U, V \in H$ that

$$\begin{aligned}
\mathrm{Ric}^\eta(U,V) &= \sum_{j=1}^{2m} R^\eta(U, b_j, b_j, V) \\
&= \sum_{j=1}^{2m} R^\eta(b_j, V, U, b_j) + d\eta(U,V) g(\tau b_j, b_j) + d\eta(b_j, U) g(\tau b_j, V) \\
&\qquad + d\eta(V, b_j) g(\tau U, b_j) \\
&= \mathrm{Ric}^\eta(V,U) + d\eta(U,V) \underbrace{\operatorname{tr}\tau}_{=0} + \sum_{j=1}^{2m} -2g(b_j, JU) g(b_j, \tau V) \\
&\qquad\qquad\qquad\qquad + 2g(JV, b_j) g(\tau U, b_j) \\
&= \mathrm{Ric}^\eta(V,U) - 2g(JU, \tau V) + g(\tau U, JV)
\end{aligned}$$

and thus, using symmetry of τ, skew-symmetry of J and their relation with g and each other,

$$\mathrm{Ric}^\eta(U,V) = \mathrm{Ric}^\eta(V,U) \quad \text{for any } U, V \in H. \tag{1.47}$$

Due to the failure of the symmetry $R^\eta(Y,U,V,W) = R^\eta(V,W,Y,U)$ as we have it in the case of the Levi/Civita connection, $\mathcal{R}^\eta(J\cdot, J\cdot) = \mathcal{R}^\eta$ does not hold (unlike in the Kähler case). Instead we introduce the following splitting for the restriction $\mathcal{R}^\eta_H \colon H \times H \to H$ of $\mathcal{R}^\eta$ to H:

$$\mathcal{R}^\eta_H = \mathcal{R}_- + \mathcal{R}_+,$$

where $\mathcal{R}_\pm = \pm\mathcal{R}_\pm(J\cdot, J\cdot)$ and the two parts are given by

$$\mathcal{R}^\eta_\pm(U,V) = \tfrac{1}{2}(\mathcal{R}^\eta(U,V) \pm \mathcal{R}^\eta(JU, JV)).$$

2 Spin geometry on metric contact and CR manifolds

In this section, we introduce the second main object of our study: Spin manifolds and their Dirac operators. We begin with a quick review of spin geometry before we move on to the special case of metric contact and CR manifolds, the structure of their spin bundles and the Dirac operators of their adapted connections.

2.1 Crash course spin geometry

This will be a very short introduction to spin geometry both for readers who are unfamiliar with the subject and to fix notation. The reader interested in more details will find a large literature on the subject, among which the book of Lawson and Michelsohn [LM89] is a very thorough treatment and a comprehensive reference on the subject (at least for any results known by the late eighties). A shorter introduction may be found in [Fri00] and the case of Pseudo-Riemannian manifolds is treated in [Bau81]. We begin with some algebra, namely Clifford algebras and their representations and then discuss spin structures on manifolds and their spinor bundles.

2.1.1 Clifford algebras and the spin group

Let V be a finite-dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) with a (positive-definite) scalar product q. The *Clifford algebra* of (V, q) is the $\mathbb{K}$-algebra generated by V and the unit 1 subject to the relation

$$vw + wv = -2q(v, w) \cdot 1$$

for any $v, w \in V$. In other words, it is the smallest associative algebra containing V and satisfying the above relation. It may be realised as the sum of tensor product of any degree of V with itself divided by the two-sided ideal generated by $v \otimes w + w \otimes v + 2q(v, w)1$. We write $Cl(V, q)$ for the Clifford algebra of (V, q). In particular, we write Cl_n for the Clifford algebra of $\mathbb{R}^n$ with the euclidean scalar product and $\mathbb{C}l_n$ for the Clifford algebra of $\mathbb{C}^n$ with the usual Hermitian scalar product. Note that $\mathbb{C}l_n$ is isomorphic to the complexification of Cl_n.

Consider the involution $\alpha\colon V \to V$, $\alpha(v) = -v$. As

$$\alpha(v) \cdot \alpha(v) = (-v)(-v) = v \cdot v = -2q(v, v) = -2q(\alpha(v), \alpha(v)),$$

this endomorphism lifts to $\tilde{\alpha}\colon Cl(V, q) \to Cl(V, q)$. As $\tilde{\alpha}^2 = Id$, the Clifford algebra splits into its ± 1-eigenspaces:

$$Cl(V, q) = Cl^0(V, q) \oplus Cl^1(V, q), \quad Cl^j(V, q) = \left\{ x \in Cl(V, q) \,\middle|\, \tilde{\alpha}(x) = (-1)^j x \right\}.$$

Under multiplication, we have

$$Cl^j(V, q) \cdot Cl^k(V, q) \subset Cl^{(j+k \mod 2)}(V, q),$$

and in particular, $Cl^0(V, q)$ is a subalgebra. For any $v \in V \setminus \{0\}$, we have

$$v \cdot \left(\frac{-v}{2q(v, v)} \right) = -\frac{1}{2} 2q \left(v, \frac{-v}{q(v, v)} \right) = 1,$$

i.e. any $v \in V \setminus \{0\}$ has an inverse. Building on this, we find the following multiplicative groups in $Cl(V, q)$.

Definition. The *pin group* $Pin(V, q) \subset Cl(V, q)$ is the group multiplicatively generated by elements $v \in V$, $q(v, v) = 1$. The *spin group* $Spin(V, q)$ is the intersection $Pin(V, q) \cap Cl^0(V, q)$.

In particular, we write Pin_n and $Spin_n$ for the pin and spin groups of Cl_n.

From the definition of $Cl^0(V, q)$ we see that $Spin(V, q)$ is multiplicatively generated by elements of the type $v \cdot w$, $v, w \in V$, $q(v, v) = q(w, w) = 1$. It is closely related to the special orthogonal group.

Lemma 2.1.1 ([LM89, Thm I.2.10])**.** *Let $n \geq 3$. Then, $Spin_n$ is the universal covering of the special orthogonal group SO_n. The covering $\lambda\colon Spin_n \to SO_n$ is two-fold.*

We now discuss representations of Clifford algebras and the representations they induce on the spin group because these will allow us to define spinor bundle on manifolds. The representation theory of the Clifford algebras $\mathbb{C}l_n$ is rather easy.

Proposition 2.1.2 ([LM89, Thms I.5.8])**.** *Let $m \in \mathbb{N}$. Then, we have the following identities for the complex Clifford algebras:*

$$\mathbb{C}l_{2m} \simeq \operatorname{End}(\mathbb{C}^{2^m}), \quad \mathbb{C}l_{2m+1} \simeq \operatorname{End}(\mathbb{C}^{2^m}) \oplus \operatorname{End}(\mathbb{C}^{2^m}).$$

Moreover, the natural representation of $\operatorname{End}(\mathbb{C}^{2^m})$ *is, up to equivalence, the only irreducible representation of the algebra $\mathbb{C}l_{2m}$. The natural representation of* $\operatorname{End}(\mathbb{C}^{2^m})$ *composed with the projection onto the first or second component are, up to equivalence, the only irreducible algebra representations of $\mathbb{C}l_{2m+1}$.*

Restricting one of the irreducible representations of $\mathbb{C}l_n$ to the spin group $Spin_n \subset Cl_n \subset \mathbb{C}l_n$ gives a group representation

$$\kappa_n\colon Spin_n \to \Delta_n.$$

This representation is called the *(complex) spin representation*. There is also a real spin representation, but we will not consider it in this thesis and therefore just speak of the spin representation. Note that as vector spaces $\Delta_{2m} = \Delta_{2m+1} = \mathbb{C}^{2^m}$. The choice of one of the two irreducible representations in the case n odd does not matter as the following proposition shows.

Proposition 2.1.3 ([LM89, Thm 1.5.15])**.** *The two irreducible representations of $\mathbb{C}l_{2m+1}$ are equivalent when restricted to $Spin_{2m+1}$ and the restriction is irreducible. When we restrict the irreducible representation of $\mathbb{C}l_{2m}$ to $Spin_{2m}$, it splits into two irreducible representations:*

$$\Delta_{2m} = \Delta_{2m}^+ \oplus \Delta_{2m}^-.$$

We will now explicitly describe a realisation of the isomorphism between the Clifford algebra and the space of matrices as we will need this later on (cf [BFGK91, Section 1.1]): We consider the vectors

$$u(\delta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\delta i \end{pmatrix} \in \mathbb{C}^2$$

and the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.1}$$

We give $\Delta_{2m+1} = \mathbb{C}^{2^m} \simeq \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ the following basis:

$$\left\{u(\delta_1, \ldots, \delta_m) = u(\delta_1) \otimes \cdots \otimes u(\delta_m)\right\}_{\delta_j = \pm 1}.$$

We will sometimes write u_δ instead of $u(\delta)$. We can then describe the representation κ as follows: Given an orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ ($n = 2m, 2m+1$), we define

$$\kappa(e_{2j-1}) = I \otimes \cdots \otimes I \otimes U \otimes \underbrace{T \otimes \cdots \otimes T}_{j-1}, \qquad j = 1, ..., m \tag{2.2}$$

$$\kappa(e_{2j}) = I \otimes \cdots \otimes I \otimes V \otimes \underbrace{T \otimes \cdots \otimes T}_{j-1} \text{ and} \tag{2.3}$$

$$\kappa(e_{2m+1}) = iT \otimes \cdots \otimes T, \tag{2.4}$$

where $\kappa(e_{2m+1})$ obviously does not appear for $n = 2m$. We want to compute the Clifford action. We have

$$Uu(\delta) = iu(-\delta), \quad Vu(\delta) = \delta u(-\delta), \text{ and } \quad Tu(\delta) = -\delta u(\delta). \tag{2.5}$$

and thus

$$\kappa(e_{2j-1})u_{\delta_1,\dots,\delta_m} = i(-1)^{j-1}\delta_{m-j+2}\cdots\delta_m u_{\delta_1,\dots,\delta_{m-j},-\delta_{m-j+1},\delta_{m-j+2},\dots,\delta_m} \tag{2.6}$$
$$\kappa(e_{2j})u_{\delta_1,\dots,\delta_m} = (-1)^{j-1}\delta_{m-j+1}\cdots\delta_m u_{\delta_1,\dots,\delta_{m-j},-\delta_{m-j+1},\delta_{m-j+2},\dots,\delta_m} \tag{2.7}$$
$$\kappa(e_{2m+1})u_{\delta_1,\dots,\delta_m} = i(-1)^m\delta_1\cdots\delta_m u_{\delta_1,\dots,\delta_m} \tag{2.8}$$

Before we finish this sections, we briefly discuss the complex spin group $Spin^c$. To define this group, we consider $Spin_n \subset \mathbb{C}l_n \simeq Cl_n \otimes \mathbb{C}$ and define

$$Spin_n^c = Spin_n \times_{\mathbb{Z}_2} S^1.$$

This group is a two-fold covering of $SO_n \times S^1$ and the covering map is given by

$$\lambda^c([g,z]) = [\lambda(g), z^2].$$

Moreover, the following diagram commutes (cf [Fri00, section 1.6]), where the arrows without labels stand for the inclusion maps and $sq(z) = z^2$.

$$\begin{tikzcd}
Spin_n \arrow[rr] \arrow[d, "\lambda"] & & Spin_n^c \arrow[dll, "pr_1\circ\lambda^c"] \arrow[drr, "pr_2\circ\lambda^c"] & & S^1 \arrow[ll] \arrow[d, "sq"] \\
SO_n & & & & S^1
\end{tikzcd}$$

The unitary group can be embedded in $Spin^c$ in a way that is compatible with the inclusion of the unitary in the special orthogonal group and the covering of SO by spin. More precisely, we have the following result.

Lemma 2.1.4 ([Fri00, section 1.6])**.** *Let $n \in \{2m, 2m+1\}$. There is a group homomorphism $F\colon U_m \to Spin_n^c = Spin_n \times_{\mathbb{Z}_2} S^1$ that makes the following diagram commutative:*

$$\begin{tikzcd}
 & Spin_n^c \arrow[d, "\lambda^c"] \\
U_m \arrow[ur, "F"] \arrow[r, "(id,\det)"'] & SO_n \times S^1
\end{tikzcd} \tag{2.9}$$

In fact, this map may be explicitly given as follows: Let $b_1, \ldots, b_m$ be a unitary basis of $\mathbb{C}^m$ such that $Ab_j = e^{i\theta_j} b_j$. Then, F can be realised as

$$F(A) = \left[\prod_{j=1}^{m} \left(\cos \tfrac{\theta_j}{2} + \sin \tfrac{\theta_j}{2} b_j J_0 b_j \right), e^{\frac{i}{2} \sum_{j=1}^{m} \theta_j} \right]. \tag{2.10}$$

This closes the algebraic discussion of Clifford algebras and spin representations. In the next section, we will transfer these structures to manifolds.

2.1.2 Spin manifolds

We begin with a short discussion of frame bundles. Associated with any n-dimensional manifold is a principal GL_n-bundle $P_{GL}(M)$, pointwise given by

$$(P_{GL}(M))_p = \{ (b_1, \ldots, b_n) \mid \text{ basis of } T_pM \} .$$

The projection is the obvious one. If the manifold is equipped with a Riemannian metric, we can reduce the structure group to O_n via

$$(P_O(M))_p = \{ (b_1, \ldots, b_n) \mid \text{ orthonormal basis of } T_pM \}$$

and further to SO_n if the Riemannian manifold is oriented:

$$(P_{SO}(M))_p = \{ (b_1, \ldots, b_n) \mid \text{ oriented orthonormal basis of } T_pM \} .$$

Now, a spin structure is a further reduction of structure group of the frame bundle.

Definition. Let (M^n, g) be a Riemannian manifold. A *spin structure* is a is a $Spin_n$-principal bundle $\pi_S \colon P_{Spin}(M) \to M$ and a two-fold covering map $f \colon P_{Spin}(M) \to P_{SO}(M)$ that is a bundle morphism and$Spin_n$-equivariant, i.e. $f(bh) = f(b)\lambda(h)$ for all $b \in P_{Spin}(M)$ and $h \in Spin_n$.

If (M, g) admits a spin structure, it is called spin.

Depending on the manifold, there may be more than one spin structure. If we speak of a spin manifold in the sequel, we assume that we have fixed a spin structure.

While the definition of the Spin structure involves a Riemannian structure on the manifold, both the existence of a spin structure and the number of spin structures on a given manifold do not depend on the metric.

Theorem 2.1.5 ([LM89, Thm II.2.1])**.** *Let (M, g) be an oriented Riemannian manifold. Then M admits a spin structure if and only if its second Stiefel-Whitney class vanishes. In that case, the equivalence classes of spin structures are in one-to-one correspondence with $H^1(M, \mathbb{Z}_2)$.*

Now, using the spin representation, we can associate a vector bundle with fibre Δ_n to a spin manifold.

Definition. Let (M^n, g) be spin. Then, the *spinor bundle* is the vector bundle

$$\mathbb{S} = P_{Spin}(M) \times_\kappa \Delta_n$$

associated with the spin principal bundle.

As the spin representation was a restriction from a representation of $\mathbb{C}l_n$, we obtain a map

$$\begin{array}{rccc} cl\colon & \mathbb{C}l_n \times \Delta_n & \longrightarrow & \Delta_n \\ & (x, v) & \longmapsto & \kappa(x)(v), \end{array}$$

which we will call Clifford multiplication. As it is *Spin*-invariant, it extends to the to the Clifford bundle

$$Cl(M, g) = \coprod_{p \in M} Cl(T_pM, g_p)$$

and the spinor bundle, i.e. we obtain a fibrewise Clifford product

$$\begin{array}{rccc} cl\colon & Cl(M, g) \times \mathbb{S} & \longrightarrow & \mathbb{S} \\ & (X, \varphi) & \longmapsto & \kappa(X)(\varphi). \end{array}$$

In particular, we have the Clifford product of tangent vectors that we will also simply write $X \cdot \varphi = cl(X)(\varphi)$. The Clifford action extends to differential forms as follows: Given a local ON basis $(b_1, \ldots, b_n)$ and its dual $(b^1, \ldots, b^n)$, we locally write $\omega \in \Omega^k(M)$ as

$$\omega = \sum_{I=(i_1<\cdots<i_k)} \omega_I b^{i_1} \wedge \cdots \wedge b^{i_k}$$

and then set

$$\omega \cdot \varphi = \sum_{I=(i_1<\cdots<i_k)} \omega_I b_{i_1} \cdots b_{i_k} \cdot \varphi \tag{2.11}$$

Alternatively, one can use the fact that $Cl_n \simeq \bigoplus_{k=0}^n \Lambda^k(\mathbb{R}^n)^*$ to obtain the same product.

There is a positive definite Hermitian scalar product on the spinor module Δ_n that makes Clifford multiplications by vectors skew-symmetric, i.e.

$$\langle x \cdot v, w \rangle = -\langle v, x \cdot w \rangle$$

for any $x \in \mathbb{R}^n$ and $v, w \in \Delta_n$. This scalar product is *Spin*-invariant and therefore induces a bundle metric on $\mathbb{S}$. The Clifford action of differential forms of degree k is then skew-symmetric if $k \equiv 1, 2 \mod 4$ and symmetric if $k \equiv 3, 4 \mod 4$.

The Hermitian scalar product extends to an L^2-product

$$(\varphi,\psi)_{L^2} = \int_M \langle \varphi(x), \psi(x)\rangle dM(x)$$

for $\varphi, \psi \in \Gamma_c(\mathbb{S})$ and one on the whole bundle of smooth sections if the manifold is compact.

2.2 Connections and Dirac operators

In order to do analysis on the spinor bundle, we need a connection. Any metric connection on (M, g) induces a connection on the spinor bundle. In this section, we review how these induced connections are defined and discuss their properties. Then, each of these connections defines a first-order differential operator, the Dirac operator. We introduce these operators and discuss their basic properties. We consider general metric connections in this section and discuss the special properties of adapted connections on metric contact manifolds in the following one.

Given a metric covariant derivative $\nabla\colon \Gamma(TM) \to \Gamma(T^*M \otimes TM)$, we can define the associated connection form $C^\nabla \in \Omega^1(P_{SO}(M), \mathfrak{so}_n)$ as follows. A connection form on a bundle P is fully defined by its local connection forms $C^b = C \circ db$, where $b \in \Gamma(U, P|_U)$ is a local section. For a local section $b = (b_1, \dots, b_n) \in \Gamma(U, P_{SO}(M)|_U)$, we set

$$(C^\nabla)^b(X) = \sum_{j<k} g(\nabla_X b_j, b_k) E_{jk},$$

where $E_{jk} \in \mathfrak{so}_n$ is the matrix given by $(E_{jk})_{\alpha\beta} = -\delta_{j\alpha}\delta_{k\beta} + \delta_{j\beta}\delta_{k\alpha}$. Now, given a spin structure $f\colon P_{Spin}(M) \to P_{SO}(M)$, there exists a lift $\widetilde{C^\nabla}$ such that the following diagram commutes.

$$\begin{tikzcd} TP_{Spin}(M) \arrow[r, "\widetilde{C^\nabla}"] \arrow[d, "df"'] & \mathfrak{spin}_n \arrow[d, "\lambda_*"] \\ TP_{SO}(M) \arrow[r, "C^\nabla"] & \mathfrak{so}_n \end{tikzcd} \tag{2.12}$$

Then, this connection form induces a covariant derivative $\nabla\colon \Gamma(\mathbb{S}) \to \Gamma(T^*M \otimes \mathbb{S})$ in the usual way. From the general formula for such connections, we see that locally, we have

$$\nabla_X \varphi|_U = [\tilde{b}, X(v) + \tfrac{1}{2}\sum_{j<k} g(\nabla_X b_j, b_k) s_j \cdot s_k \cdot v], \tag{2.13}$$

where $b \in \Gamma(U, P_{SO}(M)|_U)$, $\tilde{b}$ is its lift to $P_{Spin}(M)$, (s_j) the standard basis of $\mathbb{R}^n$ (i.e. $b_j = [b, s_j]$) and $v \in C^\infty(U, \Delta_n)$. We collect some well-known properties of this connection.

Lemma 2.2.1. *Let (M, g) be spin and let ∇ be a metric connection. Then, the connection induced on the spinor bundle is metric with respect to the Hermitian bundle metric and for any $X, Y \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(\mathbb{S})$, we have*

$$\nabla_X(Y \cdot \varphi) = \nabla_X(Y) \cdot \varphi + Y \cdot (\nabla_X \varphi). \tag{2.14}$$

Extending ∇ to forms in the usual way, an analogous result holds for the Clifford product with forms, i.e. for any $\omega \in \Omega^*(M)$, $X \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(\mathbb{S})$, we have

$$\nabla_X(\omega \cdot \varphi) = \nabla_X(\omega) \cdot \varphi + \omega \cdot (\nabla_X \varphi). \tag{2.15}$$

Proof. This is proven for the Levi Civita connection ∇^g in [Fri00, section 3.1]; the proof carries over without any changes to a general metric connection. The extension to forms follows immediately from (2.11). □

We want to compare the spinor connection induced by an adapted connection with the well-known spinor connection induced by the Levi-Civita connection. Using formulae (1.29) and (2.13), we deduce that

$$\nabla_X \varphi = \nabla^g_X \varphi - T(X; \cdot, \cdot) \cdot \varphi + \frac{3}{2} \mathfrak{b} T(X, \cdot, \cdot) \cdot \varphi. \tag{2.16}$$

We want to compare the curvature tensor of $\mathbb{S}$ with the one on TM. The following formula is well-known for the Levi Civita connection (cf e.g. [LM89, Thm II.4.15]) and we extend it to any metric connection.

Lemma 2.2.2. *Let (M, g) be a spin manifold and $\mathbb{S}$ its spinor bundle. Let ∇ be a metric connection on TM, R^∇ its (4,0)-curvature tensor and $R^{\mathbb{S}}$ the curvature tensor of the connection induced on $\mathbb{S}$ by ∇. Then, we have the following relationship between the two curvature tensors:*

$$R^{\mathbb{S}}(X, Y)\varphi = \frac{1}{4} \sum_{j,k=1}^{n} R^\nabla(X, Y, b_j, b_k) b_j \cdot b_k \cdot \varphi,$$

where (b_j) is a local ON basis of TM.

Proof. The proof is essentially the same as for the Levi-Civita connection. Let (b_j) be p-synchronous for ∇, i.e. $(\nabla b_j)(p) = 0$ and let $\mathbb{S}$ be locally trivialised by the lift to $P_{Spin}(M)$ of the local section (b_j) in $P_{SO}(M)$. In this proof we write $\nabla^{\mathbb{S}}$ for the induced connection on the spinor bundle. Then, locally,

$$\nabla^{\mathbb{S}}_X \varphi = X(\varphi) + \frac{1}{2} \sum_{j<k} g(\nabla_X b_j, b_k) b_j \cdot b_k \cdot \varphi.$$

Deriving once more, evaluating at p and using that (b_j) is p-synchronous, we obtain

$$\nabla^{\mathbb{S}}_X \nabla^{\mathbb{S}}_Y \varphi = X(Y(\varphi)) + \frac{1}{2} \sum_{j<k} X(g(\nabla_Y b_j, b_k)) b_j \cdot b_k \cdot \varphi.$$

Thus, we have

$$\begin{aligned} R^{\mathbb{S}}(X,Y)\varphi &= X(Y(\varphi)) - Y(X(\varphi)) - [X,Y](\varphi) \\ &\quad + \frac{1}{2} \sum_{j<k} \big(X(g(\nabla_Y b_j, b_k)) - Y(g(\nabla_X b_j, b_k)) - g(\nabla_{[X,Y]} b_j, b_k) \big)\, b_j \cdot b_k \cdot \varphi \\ &= \frac{1}{2} \sum_{j<k} \left(X(g(\nabla_Y b_j, b_k)) - Y(g(\nabla_X b_j, b_k)) \right) b_j \cdot b_k \cdot \varphi. \end{aligned} \tag{2.17}$$

On the other hand, we have

$$\nabla_X b_k = \sum_{j=1}^{n} g(\nabla_X b_k, b_j) b_j$$

and thus, at $p \in M$,

$$\begin{aligned} \nabla_X \nabla_Y b_k &= \sum_{j,1}^{n} X(g(\nabla_Y b_k, b_j)) b_j, \\ R^{\nabla}(X,Y,b_k,b_j) &= \sum_{l=1}^{n} g\left(\big(X(g(\nabla_Y b_k, b_l)) - Y(g(\nabla_X b_k, b_l)) - g(\nabla_{[X,Y]} b_k, b_l) \big)\, b_l, b_j \right) \\ &= X(g(\nabla_Y b_k, b_j)) - Y(g(\nabla_X b_k, b_j)). \end{aligned}$$

Comparing this with (2.17) and using skew-symmetry in X, Y yields the claim. □

Associated with every spinor connection is a first-order elliptic differential operator, the Dirac operator.

Definition. Let (M,g) be spin and let ∇ be a metric connection on M. Then, the first-order differential operator

$$D^{\nabla} : \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathbb{S}) \xrightarrow{cl} \Gamma(\mathbb{S})$$

is called the Dirac operator associated with ∇. We write D^g for the Dirac operator associated with the Levi-Civita connection and D^η for the one associated with the Tanaka-Webster connection.

Locally, the Dirac operators is given by

$$D^\nabla \varphi = \sum_{j=1}^{n} b_j \cdot \nabla_{b_j} \varphi,$$

where (b_j) is a local orthonormal frame. An easy calculation shows that

$$D^\nabla (f \cdot \varphi) = f D^\nabla \varphi + \operatorname{grad} f \cdot \varphi \tag{2.18}$$

for $f \in C^\infty(M)$ and $\varphi \in \Gamma(\mathbb{S})$.

The Dirac operator D^g is a well-studied object (compare [Gin09] for a collection of results, with an emphasis on the spectrum) and we will use it as a point of reference and compare the other Dirac operators with it. Using the local formula for the Dirac operator and the comparison formula (2.16), one obtains that

$$D^\nabla = D^g + \frac{1}{2} \operatorname{tr} T + \frac{3}{4} \mathfrak{b} T. \tag{2.19}$$

From this, one deduces the following well-known fact, originally (in a different formulation) due to Friedrich and Sulanke [FS79], compare also [Nic05, Prop 1.6].

Lemma 2.2.3. *Let (M, g) be spin and ∇ a metric connection of torsion T. Then, the Dirac operator D^∇ is formally self-adjoint, i.e. $(D^\nabla \varphi, \psi)_{L^2} = (\varphi, D^\nabla \psi)_{L^2}$ for $\varphi, \psi \in \Gamma_c(\mathbb{S})$, if and only if $\operatorname{tr} T = 0$.*

Proof. This follows from (2.19) because multiplication by a three-form is symmetric, multiplication by a one-form is skew-symmetric and the Dirac operator D^g is formally self-adjoint. □

The original formulation was that D^∇ is formally self-adjoint if and only if the divergence of ∇

$$\operatorname{div}^\nabla(X) = \sum_{j=1}^{n} g(\nabla_{b_j} X, b_j),$$

where (b_j) is a local orthonormal basis, is equal to the usual divergence div^g, i.e. that of ∇^g. One may show directly that this condition is equivalent to the vanishing of the trace.

Lemma 2.2.4. *Let (M, g) be a spin manifold and ∇ a metric Then, we have $\operatorname{div}^\nabla = \operatorname{div}$ if and only if the torsion of ∇ is traceless.*

Proof. For any $X \in \mathfrak{X}(M)$ and a local orthonormal basis (b_j) of TM, we have

$$\begin{aligned} \operatorname{div}^\nabla(X) &= \sum_{j=1}^{n} g(\nabla_{b_j} X, b_j) \\ &= \sum_{j=1}^{n} g(\nabla^g_{b_j} X, b_j) - T(b_j; X, b_j) + \tfrac{3}{2} \mathfrak{b} T(b_j, X, b_j). \end{aligned}$$

As $\flat T$ is skew-symmetric, the last term vanishes. Summing over the second term yields $\operatorname{tr} T$ and the result follows. $\square$

Note that formal self-adjointness is not the same self-adjointness, i.e. the adjoint operator

$$(D^\nabla)^*\colon \operatorname{dom}((D^\nabla)^*) \subset L^2(\mathbb{S}) \to L^2(\mathbb{S})$$

of $D^\nabla\colon \Gamma(\mathbb{S}) \subset L^2(\mathbb{S}) \to L^2(\mathbb{S})$ does not necessarily coincide with D^∇. As D^∇ is symmetric, they agree where both are defined but $\operatorname{dom}((D^\nabla)^*)$ may be larger than $\Gamma(\mathbb{S})$. In fact, it is larger but we will see that if we move from D^∇ to its closure, we obtain a self-adjoint operator. The closure of D^∇ is the linear operator $\overline{D^\nabla}$ defined on the space of all spinors $\varphi \in L^2(\mathbb{S})$ for which there exists a $\psi \in L^2(\mathbb{S})$ such that for any sequence $(\varphi_n) \subset L^2(\mathbb{S})$ that converges to φ, the sequence $(D^\nabla \varphi_n)$ converges to ψ. Then, $\overline{D^\nabla}\varphi = \psi$.

Lemma 2.2.5 ([Sta12, Lemma 5.4])**.** *Let (M, g) be spin and complete and ∇ a metric connection such that D^∇ is symmetric. Then, D^∇ is essentially self-adjoint, i.e. its closure is a self-adjoint operator.*

Proof. Consider the proof of essential self-adjointness of the Dirac operator of Wolf, cf. [Wol72]. The proof is given only for D^g, but is indeed extendable to all symmetric Dirac operators: The domain of D^∇ is $\Gamma_c(\mathbb{S}) \subset L^2(\mathbb{S})$. The proof uses the following norm on the domain of $(D^\nabla)^*$:

$$N(\varphi) = \sqrt{\|\varphi\|^2_{L^2} + \|(D^\nabla)^*\varphi\|^2_{L_2}}.$$

Then the following results are proven:

1. If $\Gamma_c(\mathbb{S})$ is dense in $\operatorname{dom}((D^\nabla)^*)$ with respect to the N norm, then $(D^\nabla)^*$ is essentially self-adjoint.

2. $\Gamma_c(\mathbb{S})$ is dense in $\operatorname{dom}_c((D^\nabla)^*)$ with respect to the N norm, where dom_c denotes all elements of the domain with compact support.

3. If (M, g) is complete, then $\operatorname{dom}_c((D^\nabla)^*)$ is dense in $\operatorname{dom}((D^\nabla)^*)$ with respect to the N norm.

The proof of the first statement requires nothing of D^∇ but to be a closable operator. To prove the second one, we only need D^∇ to be an elliptic differential operator of order one, which it is because it differs form D^g only by lower order terms. Finally, the proof of the third statement does not make use of the explicit form of D^∇ either, it only needs it to fulfil the product rule (2.18). $\square$

Note that if we assume the manifold not only complete, but closed (i.e. compact without boundary), the above result is true for any formally self-adjoint elliptic operator, compare for example [Shu01, Theorem 8.3].

The Dirac operators of two metric connections are not necessarily different. In fact, one easily deduces the following fact from (2.19).

Lemma 2.2.6 ([Nic05, Equation (1.5)])**.** *Let (M, g) be spin and ∇^j $(j = 1, 2)$ metric connections of torsion T_j. Then, $D^{\nabla^1} = D^{\nabla^2}$ if and only if $\operatorname{tr} T^1 = \operatorname{tr} T^2$ and $\mathfrak{b}T^1 = \mathfrak{b}T^2$.*

Proof. The trace and $\mathfrak{b}T$ both have to be equal because the Clifford action of two forms is equal if and only if the forms are equal and $\operatorname{tr} T$ and $\mathfrak{b}T$ are of different degree. □

We will call two connections that induce the same Dirac operator *Dirac-equivalent.*

2.3 The spinor bundle over metric contact manifolds

Just like the spinor bundle over a Kähler manifold, the spinor bundle over a metric contact manifold (and thus, in particular, over a strictly pseudoconvex CR manifold) splits into subbundles that are eigenspaces of the Kähler form $F = g(J\cdot, \cdot)$, or equivalently, of $d\eta$.

We begin by noting the following: The SO-frame bundle of a metric contact manifold may be reduced to a U-bundle:

$$P_U(M)_p = \{(e_1, Je_1, \ldots, e_m, Je_m, \xi) \; ON - \text{basis of } T_pM\} \subset P_{SO}(M)_p.$$

Now, given a spin structure $f\colon P_{Spin}(M) \to P_{SO}(M)$, we can reduce the $Spin$-principal bundle to a $\lambda^{-1}(U)$-principal bundle (where $\lambda\colon Spin \to SO$ is the two-fold covering) via

$$P^H_{Spin}(M) = f^{-1}(P_U(M)), \quad \iota\colon P^H_{Spin}(M) \to P_{Spin}(M),$$

where ι is the inclusion map. Then, for the spinor bundle we have

$$\mathbb{S} = P_{Spin}(M) \times_\kappa \Delta_{2m+1} = P^H_{Spin}(M) \times_{\kappa\circ\iota} \Delta_{2m+1}. \tag{2.20}$$

Because TM, the bundle of forms and the Clifford bundle may be similarly associated with $P^H_{Spin}(M)$, the Clifford multiplication remains unchanged.

Given an adapted connection $\nabla\colon \Gamma(TM) \to \Gamma(T^*M \otimes TM)$, the connections stabilises H and commutes with J and thus, the induced connection form C^∇ may be reduced to a connection on $P_U(M)$. Then, the covariant derivative defined on $\mathbb{S}$ via P_U and P^H_{Spin} will be the same as the one defined in the usual way.

Proposition 2.3.1 ([Bau99, Prop 22]). *Let (M, g, J, η) be a spin metric contact manifold. Then, the spinor bundle $\mathbb{S}$ splits under the Clifford action of the Kähler form F into*

$$\mathbb{S} = \bigoplus_{k=0}^{m} \mathbb{S}_{(m-2k)} \tag{2.21}$$

where

$$\begin{aligned} \mathbb{S}_{(m-2k)} &= \{\varphi \in \mathbb{S} \mid F \cdot \varphi = (m-2k)i\varphi\} \\ &= \{\varphi \in \mathbb{S} \mid d\eta \cdot \varphi = 2(m-2k)i\varphi\}. \end{aligned}$$

Proof. Let $(e_1, Je_1, \ldots, e_m, Je_m)$ be a local basis of $\mathbb{R}^{2m}$, where J is induced by the multiplication with i on $\mathbb{R}^{2m} \simeq \mathbb{C}$. Considering (2.20), we can locally write $F|_V = [\tilde{s}, \Omega]$, where $\tilde{s}$ is the lift of an adapted frame $s \in \Gamma(V, P_U(M))$ and thus,

$$\Omega = e_1 \wedge Je_1 + \cdots + e_m \wedge Je_m.$$

From (2.5), we obtain that

$$\Omega \cdot u(\delta_1, \ldots, \delta_m) = \left(i \sum_{j=1}^{m} \delta_j \right) u(\delta_1, \ldots, \delta_m)$$

and thus, Δ_{2m+1} splits into the eigenspaces $\Delta_{i(m-2k)} = E(\Omega, i(m-2k))$ and a basis of $\Delta_{(m-2k)i}$ is given by

$$\{u(\delta_1, \ldots, \delta_m) \mid \#\{j | \delta_j = -1\} = k\}.$$

It remains to show that these subspaces are invariant under the action of $\lambda^{-1}(U)$. To see this, we need to know more about the elements of the Lie group $\lambda^{-1}(U)$. Recall the formula (2.10) for the mapping $F \colon U_m \to Spin^c_{2m+1}$ that makes diagram (2.9) commute. Now let $A \in U_m$. From the commutativity of the diagram we see that the first component of $F(A) = [g(A), z(A)]$ is in $\lambda^{-1}(A)$. Furthermore, we have

$$\lambda^c([-g(A), z(A)]) = (\lambda, sq)([g(A), -z(A)]) = (\lambda(g(A)), (-z(A))^2) = (A, \det(A))$$

and thus $\lambda^{-1}(A) = \{\pm g(A)\}$. From the form of $g(A)$, we see that $\Delta_{(2m-k)}$ is $\lambda^{-1}(U)$-invariant and thus, the splitting carries over to the bundle

$$\mathbb{S} = \bigoplus_{k=0}^{m} \mathbb{S}_{(m-2k)}.$$

$\square$

We now consider spinor connections and Dirac operators associated with adapted connections. For an adapted connection, one might use (2.16) to produce a formula for the spinor connection in terms of the different parts of the torsion. However, this formula would be rather long and uninstructive. We will later produce such a formula for Dirac operators. Instead, we consider how the splitting of the spinor bundle into eigenspaces of $d\eta$ behaves under an adapted connection.

Lemma 2.3.2. *Let (M, g, J, η) be a spin metric contact manifold and ∇ an adapted connection. Then, ∇ stabilises the splitting into eigenspaces of $d\eta$ of the spinor bundle $\mathbb{S}$, i.e. for any $X \in \mathfrak{X}(M)$ we have*

$$\nabla_X(\Gamma(\mathbb{S}_j)) \subset \Gamma(\mathbb{S}_j).$$

Proof. This follows immediately from (2.15) and the fact that $d\eta$ is parallel under ∇ (cf Lemma 1.4.1). □

This is basically all there is to say for spinor connections of adapted connections and we now turn our attention to Dirac operators. The following discussion is taken from the author's published paper [Sta12].

We will use the formulae in the preceding section to establish some properties of Dirac operators associated with adapted connections. To this end, we calculate the trace of the torsion of an adapted connection and its image under the Bianchi operator. Recall (equation (1.30)) that the torsion of such a connection is given by

$$T = N^{0,2} + \tfrac{9}{8}\omega - \tfrac{3}{8}\mathfrak{M}\omega + B + \xi \otimes d\eta - \tfrac{1}{2}\eta \wedge (J\mathcal{J}) + \eta \wedge \Phi,$$

where we can freely choose

$$\begin{aligned}
&\omega \in \Omega^+(H) = \{\omega \in \Omega^3(H) \mid \omega = \omega^{2,1} + \omega^{1,2} \text{ as (p,q)-forms}\},\\
&B \in \Omega^{1,1}_s(H, H) = \{D \in \Omega^2(H, H) \mid D(J\cdot, \cdot) = D \text{ and } \mathfrak{b}D = 0\},\\
&\Phi \in \mathrm{End}^J_-(H) = \{F : H \to H \mid g(X, FY) = -g(FX, Y) \text{ and } FJ = JF\}.
\end{aligned}$$

We know that ω and $N^{0,2}$ are traceless and so is $\eta \wedge \Phi$ because Φ is skew-symmetric. Furthermore, we immediately see that $\mathrm{tr}(\eta \wedge (J\mathcal{J}))(X) = 0$ and we calculate for some adapted basis (e_i, f_i), making use of the various properties of $\mathcal{J}$:

$$\begin{aligned}
\mathrm{tr}(\eta \wedge (J\mathcal{J}))(\xi) &= \sum_{j=1}^{m} (\eta \wedge J\mathcal{J})(e_j, e_j, \xi) + (\eta \wedge J\mathcal{J})(f_j, f_j, \xi)\\
&= \sum_{j=1}^{m} g(e_j, J\mathcal{J}e_j) + g(f_j, J\mathcal{J}f_j)\\
&= 0.
\end{aligned}$$

Thus, we deduce

$$\operatorname{tr} T = -\tfrac{3}{8} \operatorname{tr} \mathfrak{M}\omega + \operatorname{tr} B. \tag{2.22}$$

Concerning the Bianchi operator, recall formula 1.44:

$$\mathfrak{b}T = \omega + \tfrac{1}{3}\eta \wedge d\eta + \mathfrak{b}(\eta \wedge \Phi).$$

Using these equations, we deduce the following result.

Proposition 2.3.3. *The Dirac operators of adapted connections on an α-metric contact manifold have the following properties:*

(1) *The Dirac operator D^∇ associated with the adapted connection $\nabla = \nabla(\omega, B, \Phi)$ is formally self-adjoint if and only if $\operatorname{tr}(B) = \frac{3}{8} \operatorname{tr} \mathfrak{M}\omega$. Moreover, if (M, g) is complete, the Dirac operator of any such connection is essentially self-adjoint.*

(2) *Two adapted connections $\nabla(\omega, B, \Phi)$ and $\nabla(\hat{\omega}, \hat{B}, \hat{\Phi})$ whose Dirac operators are formally self-adjoint are Dirac-equivalent if and only if $\omega = \hat{\omega}$ and $\Phi = \hat{\Phi}$.*

   Thus, any Dirac equivalence class of adapted connections with formally self-adjoint Dirac operators is determined by ω, Φ, while the connections in it are parametrised by $B \in \Omega^{1,1}_s(H, H)$ such that $\operatorname{tr} B = \frac{3}{8} \operatorname{tr} \mathfrak{M}\omega$.

Proof. The first part of (1) is obvious from (2.22). For (2), recall that $\mathfrak{b}(\eta \wedge \Phi)$ completely determines Φ (compare lemma 1.3.4). □

Remark. In particular, we see that in a the Dirac equivalence class of connections inducing a formally self-adjoint Dirac operator, there is a CR connection if and only if all connections in this class are CR. Thus, contrary to the claim in the last section of [Nic05], there may be more than one CR connection in a Dirac equivalence class, as B may still be chosen freely as long as it satisfies $\operatorname{tr} B = 0$ (due to (1) of the above proposition and $\omega = 0$). In fact, the uniqueness proof in the above paper uses that the torsion of a CR connection would satisfy $T(X; Y, Z) = 0$ for any $X, Y, Z \in H$, which seems to be wrong.

We now use these results to characterise some connections that are Dirac equivalent to certain known connections:

Corollary 2.3.4. *An adapted connection $\nabla(\omega, B, \Phi)$ is Dirac equivalent to the generalised Tanaka-Webster connection if and only if it satisfies $\omega = 0$, $\Phi = 0$ and* $\operatorname{tr} B = 0$*. Any such connection is CR and its Dirac operator takes the form*

$$D^\nabla = D^g + \tfrac{1}{4} cl(\eta \wedge d\eta).$$

Proof. For $D^\nabla = D^\eta$, the Dirac operator D^∇ will need to be formally self-adjoint. Thus, by proposition 2.3.3, $\operatorname{tr} B = \frac{3}{8} \operatorname{tr} \mathfrak{M}\omega$. Because all freely choosable parts of T^η vanish, we obtain, again by proposition 2.3.3, that ω and Φ vanish, which in turn implies $\operatorname{tr} B = 0$. The explicit formula is immediately deduced from the above calculations of $\mathfrak{b}T$ and from (2.19). □

Corollary 2.3.5. *An adapted connection $\nabla(\omega, B, \Phi)$ is Dirac equivalent to the Levi-Civita connection if and only if it satisfies $\omega = 0$, $\operatorname{tr} B = 0$ and $\Phi = -\frac{1}{2}J$. Any such connection is CR.*

Proof. Again, the Dirac operator will need to be formally self-adjoint, i.e. $\operatorname{tr} B = \frac{3}{8} \operatorname{tr} \mathfrak{M}\omega$. Now, for the comparison with ∇^g we cannot use proposition 2.3.3 as ∇^g is not adapted. Instead, using (1.44), we deduce the condition $0 = \omega + \frac{1}{3}\eta \wedge d\eta + \mathfrak{b}(\eta \wedge \Phi)$. As they take their arguments from different spaces, ω and $\frac{1}{3}\eta \wedge d\eta + \mathfrak{b}(\eta \wedge \Phi)$ will have to vanish separately. We calculate

$$\tfrac{1}{3}(\eta \wedge d\eta)(\xi, X, Y) = -\mathfrak{b}(\eta \wedge \Phi)(\xi, X, Y) \quad \Leftrightarrow \quad d\eta(X, Y) = -2g(Y, \Phi X).$$

Using that $d\eta = g(J\cdot, \cdot)$ then yields then claim. □

Note that we have just proven that adapted connections may induce the same Dirac operators as non-adapted ones.

The Dirac operator of the Tanaka-Webster connection is "deficient" in two ways: First, it does not stabilise the splitting $\mathbb{S} = \bigoplus_j \mathbb{S}_j$ into eigenspaces of $d\eta$. Not only does that not "seem right", it would be a major obstacle for any attempt to obtain spectral bounds for the operator. More importantly, it does not transform in a covariant way if we conformally change the contact form η (as might be guessed from the formula (1.17) for changing ξ). Any hope for finding information on the CR geometry of the manifold in the Dirac operator or its spectrum is thus essentially lost. Both problems will disappear if we instead consider a horizontal Dirac operator that derives only in the direction of H. The following chapter will be devoted to the study of such operators.

3 Horizontal Dirac operators

In this chapter, we discuss horizontal Dirac operators associated with adapted connections on metric contact and CR manifolds. These are obtained much in the same way as normal Dirac operators, but, instead of deriving in all directions, they only derive in the direction of H. For the Tanaka-Webster connection, this Dirac operator has two main advantages over the full one, making it more suited to the CR structure: It changes in a conformally covariant way when we conformally change the contact form and its square stabilises the splitting of the spinor bundle into eigenspaces of $d\eta$.

The idea of only deriving in the direction of H has already been used for the *Sub-Laplacian* of a strictly pseudoconvex CR manifold, which has already been studied in some detail. Among other results, there are lower bounds for its spectrum [Gre85, BD97] and Obata-type theorems [IV12, LW13]. This Sub-Laplacian will also turn up in our computations of the spectrum of horizontal Dirac operators in section 3.4.

In a first section, we introduce the operator and discuss its basic properties. In the second section, we provide Weitzenböck and Schrödinger-Lichnerowicz type formulae for these operators. In the next section, we prove CR-conformal covariance of the horizontal Dirac operator associated with the Tanaka-Webster connection. The two remaining sections are devoted to examples, namely S^1-bundles and quotients of the Heisenberg group.

3.1 Definition and basic properties

In this section, we introduce horizontal Dirac operators and discuss some of their properties including formal self-adjointness. We do not change the way we construct the spinor bundle or the connections on it. The difference is only in the definition of the Dirac operator itself.

Definition. Let (M, g, J, η) be a spin metric contact manifold and ∇ an adapted connection on M. Then, the first order differential operator defined by

$$D_H^{\nabla} \colon \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathbb{S}) \xrightarrow{\mathrm{proj}_\eta \otimes \mathrm{id}} \Gamma(H^* \otimes \mathbb{S}) \overset{L_\eta}{\simeq} \Gamma(H \otimes \mathbb{S}) \xrightarrow{cl} \Gamma(\mathbb{S})$$

is called the *horizontal Dirac operator* associated with ∇. We will write D_H^η for the horizontal Dirac operator associated with ∇^η and sometimes call it the Tanaka-Webster operator.

In analogy with the Kohn Laplacian on CR manifolds, the horizontal Dirac operator is sometimes called Kohn Dirac operator in the literature. Locally, this operator can be written as

$$D_H^{\nabla}\varphi = \sum_{j=1}^{2m} e_j \cdot \nabla_{e_j}\varphi,$$

where (e_j) is an orthonormal basis of H. This local formula indicates that D_H^{∇} should be independent of ξ, making it much more likely that this operator will transform covariantly if we change the contact form η. This will indeed be the case, as we will see in section 3.3.

Clearly, this operator is not elliptic anymore. In chapter 4, we will discuss a symbolic calculus adapted to such operators that will show that horizontal Dirac operators do have "nice" analytic and spectral properties. For the time being, we will stick to geometric properties and ignore the analytic problems.

We begin by discussing the formal self-adjointness (or symmetry) of the horizontal Dirac operators. For a more general statement in slightly different context, compare [KU13, Lem 2.1].

Proposition 3.1.1. *Let (M, g, J, η) be a spin metric contact manifold and ∇ an adapted connection on M with torsion T. Then, the following statements are equivalent:*

(1) *The horizontal Dirac operator D_H^{∇} is formally self-adjoint.*

(2) *The Dirac operator D^{∇} is formally self-adjoint.*

(3) *The torsion tensor is traceless:* $\operatorname{tr} T = 0$.

(4) *The divergence defined by ∇ equals the Riemannian divergence:* $\operatorname{div}^{\nabla} = \operatorname{div}^{g}$.

For an adapted connection $\nabla(\omega, B, \Phi)$ this is equivalent to the condition $\operatorname{tr}(B) = \frac{3}{8}\operatorname{tr}\mathfrak{M}\omega$.

In particular, the operator D_H^{η} associated with the Tanaka-Webster connection is formally self-adjoint.

Proof. We fix a point $p \in M$ and a p-synchronous orthonormal basis (b_j) of H for

∇, i.e. $\nabla_{b_j} b_k(p) = 0$. At p, we have

$$\begin{aligned}
\langle D_H^\nabla \varphi(p), \psi(p)\rangle &= \sum_{j=1}^{2m} \langle b_j \cdot \nabla_{b_j}\varphi(p), \psi(p)\rangle \\
&= -\sum_{j=1}^{2m} \langle \nabla_{b_j}\varphi(p), b_j \cdot \psi(p)\rangle \\
&= -\sum_{j=1}^{2m} b_j(\langle \varphi(p), b_j \cdot \psi(p)\rangle) - \langle \varphi(p), b_j \cdot \nabla_{b_j}\psi(p)\rangle \\
&= \langle \varphi(p), D_H^\nabla \psi(p)\rangle - \sum_{j=1}^{2m} b_j(\langle \varphi(p), b_j \cdot \psi(p)\rangle).
\end{aligned}$$

There is a unique vector field $V = V_{\varphi,\psi} \in \Gamma(TM_{\mathbb{C}})$ satisfying

$$g(V, W) = i\langle \varphi, W \cdot \psi\rangle$$

for any vector field $W \in \Gamma(TM)$, where we extended g complex bilinearly. Thus, again using that (b_j) is synchronous, we obtain

$$\begin{aligned}
\sum_{j=1}^{2m} b_j(\langle \varphi(p), b_j \cdot \psi(p)\rangle) &= \sum_{j=1}^{2m} b_j(g(V, b_j))(p) \\
&= \sum_{j=1}^{2m} g(\nabla_{b_j} V, b_j)(p)
\end{aligned}$$

We denote $\operatorname{div}_H^\nabla := \sum_{j=1}^{2m} g(\nabla_{b_j} V, b_j) = \operatorname{div}(V) - g(\nabla_\xi V, \xi)$ (extending all operators complex-linearly). Integrating the above equations we obtain

$$(D_H^\nabla \varphi, \psi)_{L^2} - (\varphi, D_H^\nabla \psi)_{L^2} = \int_M \operatorname{div}_H^\nabla(V) dM$$

If we consider the full Dirac operator D_H^η, using the same arguments, we obtain

$$(D^\nabla \varphi, \psi)_{L^2} - (\varphi, D^\nabla \psi)_{L^2} = \int_M \operatorname{div}^\nabla(V) dM$$

Now, if we assume D_H^∇ formally self-adjoint, this reduces to

$$(D^\nabla \varphi, \psi)_{L^2} - (\varphi, D^\nabla \psi)_{L^2} = \int_M g(\nabla_\xi V, \xi) dM. \tag{3.1}$$

Next, we write $V = U + f\xi$ with $U \in \Gamma(H)$ and $f = g(V, \xi)$ and deduce

$$\operatorname{div}^g(V) = \operatorname{div}^g(U) + f \operatorname{div}^g(\xi) + \xi(f).$$

As ξ is divergence-free (cf Corollary 1.1.7), we obtain

$$\operatorname{div}^g(V) - \operatorname{div}^g(U) = \xi(f) = \xi(g(V, \xi)) = g(\nabla_\xi V, \xi).$$

Integrating over M, $\operatorname{div}^g(V) - \operatorname{div}^g(U)$ vanishes by Stokes' theorem and thus, by (3.1), D^∇ is formally self-adjoint. If, on the other hand, we assume D^∇ formally self-adjoint, we see that

$$(D_H^\nabla \varphi, \psi)_{L^2} - (\varphi, D_H^\nabla \psi)_{L^2} = \int_M g(\nabla_\xi V, \xi) dM.$$

By the same arguments as above, this integral vanishes and D_H^∇ is formally self-adjoint. This proves that (1) and (2) are equivalent. The remaining assertions have been proven in Lemmas 2.2.3, 2.2.4 and Proposition 2.3.3. For the Tanaka-Webster connection $\omega = 0, B = 0$, so the condition is trivially satisfied. □

It will often be useful to know the following product rule for D_H^∇ acting on products of functions and spinors.

Lemma 3.1.2. *For any spinor $\varphi \in \Gamma(\mathbb{S})$ and any $f \in C^\infty(M)$, we have*

$$D_H^\nabla(f \cdot \varphi) = \operatorname{grad}_H(f) \cdot \varphi + f \cdot D_H^\nabla \varphi,$$

where grad_H is the horizontal gradient, i.e. $\operatorname{grad}_H = pr_H(\operatorname{grad}^g)$, given in a local ON basis (b_j) of H by $\operatorname{grad}_H f = \sum_{j=1}^{2m} b_j(f) b_j$.

Proof. As the spinor derivative is a covariant derivative, it satisfies the product rule $\nabla_X(f\varphi) = X(f)\varphi + f\nabla_X\varphi$. Then, the claim is immediately deduced from the local formula for D_H^∇. □

We want to obtain further product formulae for Clifford products and the square of the horizontal Dirac operator acting on products of functions and spinors. We will discuss these only in the case of D_H^η, i.e. we restrict ourselves to the Tanaka-Webster connection.

These formulae will involve sub-differentials of forms and the Sub-Laplacian, which we will now introduce. To avoid any potential problems in taking adjoints, we always assume the manifold to be closed, i.e. compact with empty boundary here.

Definition (Sub-differential, Sub-Laplacian)**.**

1. The space of horizontal differential forms of order k is
$$\Omega^k(H) = \left\{ \omega \in \Omega^k(M) \,\middle|\, \xi \lrcorner \omega = 0 \right\}.$$

2. The sub-differential $d_{H,k} \colon \Omega^k(H) \to \Omega^{k+1}(H)$ is given by
$$d_{H,k}\omega(X_0, ..., X_k) = d\omega(pr_H X_0, ..., pr_H X_k).$$

3. The sub-codifferential $d^*_{H,k} \colon \Omega^k(H) \to \Omega^{k-1}(H)$ is the formal adjoint of $d_{H,k}$ with respect to the L^2-metric on $\Omega^k(H)$.

4. The Sub-Laplacian of M is the second-order differential operator $\Delta_H \colon C^\infty(M) \to C^\infty(M)$ given by
$$\Delta_H(f) = d^*_H d_H f.$$

Remark (Notation)**.** In what follows, we will abbreviate the projection onto H in each of the $k+1$ components by pr^k_H, i.e. $d_{H,k}\omega = d\omega \circ pr^k_H$. We will also write d_H instead of $d_{H,k}$ whenever the order k is clear or not important and do likewise for $d^*_{H,k}$.

We have the following alternative definition of the Sub-Laplacian:

Proposition 3.1.3 ([DT06, section 2.1][1])**.** *The Sub-Laplacian is also given by*
$$\Delta_H(f) = -\operatorname{div}^g(\operatorname{grad}_H f) = -\operatorname{div}^\eta(\operatorname{grad}_H f).$$

As we will often work with local formulae for the Dirac operator, we also want to give local formulae for the sub-differential and the Sub-Laplacian.

Lemma 3.1.4. *Let (M, H, J, η) be a closed strictly pseudoconvex CR manifold. For $\omega \in \Omega^k(H)$ and $\alpha \in \Omega^1(H)$, the sub-differential and its adjoint are locally given by*
$$\begin{aligned} d_H\omega(X_1, ..., X_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j+1} (\nabla^\eta_{pr_H(X_j)}\omega)(pr_H(X_0), ..., \hat{X}_j, ..., pr_H(X_k)) \\ &= \sum_{j=1}^{k+1} (-1)^{j+1} (\nabla^\eta_{pr_H(X_j)}\omega)(X_0, ..., \hat{X}_j, ..., X_k), \\ d^*_H\alpha &= -\sum_{j=1}^{2m} (\nabla^\eta_{b_j}\alpha)(b_j), \end{aligned}$$
where (b_j) is a local ON basis of H and $\hat{X}_j$ means that this argument does not appear.

[1] Note the different sign conventions here and in [DT06].

Proof. The following formula for the full differential d, valid for any metric connection ∇ with torsion T, can be found in [Gau97, section 3.5]:

$$\begin{aligned} d\omega(X_1,\dots,X_{k+1}) = &\sum_{j=1}^{k+1}(-1)^{j+1}(\nabla_{X_j}\omega)(X_1,\dots,\hat{X}_j,\dots,X_{k+1}) \\ &+\sum_{j<l}(-1)^{j+l}\omega(T(X_j,X_l),X_1,\dots,\hat{X}_j,\dots,\hat{X}_l,\dots X_{k=1}) \end{aligned} \tag{3.2}$$

Choosing $\nabla = \nabla^\eta$, we see that $T(pr_H(X_j), pr_H(X_l)) = d\eta(pr_H(X_j), pr_H(X_k))\xi$. As $\omega \in \Omega^k(H)$, we have $\xi \lrcorner \omega = 0$ and we obtain the claimed formula. The projection to H can be omitted in the arguments of $\nabla^\eta_{pr_H(X_j)}\omega$ as $\omega \in \Omega^k(H)$ and H is stable under ∇^η.

Concerning the co-differential, we have the following formula for any metric connection ∇ with torsion T:

$$\begin{aligned} d^*\omega(X_1,\dots,X_{k-1}) = &-\sum_{j=1}^{2m+1}(\nabla_{b_j}\omega)(b_j,X_1,\dots,X_{k-1}) + \omega((\operatorname{tr}T)^\sharp,X_1,\dots,X_{k-1}) \\ &-\sum_{j=1}^{k-1}(-1)^j g(T(X_j;\cdot,\cdot),\omega(\cdot,\cdot,X_1,\dots,\hat{X}_j,\dots,X_{k-1}). \end{aligned}$$

For $\nabla = \nabla^\eta$, the torsion is traceless. Moreover, for a one-form, the last sum is empty. The formula for d^*_H then follows using the fact that $d^*_H\alpha = d^*\alpha$ for any $\alpha \in \Omega^1(H)$, because for any $f \in C^\infty(M)$

$$\langle d_H f, \alpha\rangle = \langle df, \alpha\rangle = \langle f, d^*_H\alpha\rangle,$$

where the first equality comes from the fact that other parts of *df* vanish when taking the scalar product with α. $\square$

Remark. Note that the formula for the full differential d in terms of ∇^η involves torsion terms (compare (3.2)), whereas the on for d_H does not. This is another point where ∇^η and the horizontal structure fit together well.

These formulae may alternatively be written as follows

$$\begin{aligned} d_H\omega &= \sum_{j=1}^{2m}(b_j)^\flat \wedge (\nabla^\eta_{b_j}\omega)\circ pr^k_H, \\ d^*_H\alpha &= -\sum_{i=1}^{2m} b_j \lrcorner (\nabla^\eta_{b_j}\alpha), \end{aligned}$$

where (b_j) is an orthonormal basis of H. We are now ready to prove the following lemma, which is an analogue of a similar result in the Riemannian case, cf. e.g. [Gin09, Lemma 1.3.3].

Lemma 3.1.5. *Let $f \in C^\infty(M)$, $X \in \Gamma(H)$ and $\varphi \in \Gamma(\mathbb{S})$, then the following identities are satisfied:*

$$D_H^\eta(X \cdot \varphi) = -X \cdot D_H^\eta \varphi - 2\nabla_X^\eta \varphi + (d_H + d_H^*)X^\flat \cdot \varphi, \tag{3.3}$$

$$(D_H^\eta)^2(f\varphi) = f(D_H^\eta)^2\varphi - 2\nabla^\eta_{\mathrm{grad}_H f}\varphi + (\Delta_H f)\varphi - \xi(f)d\eta \cdot \varphi. \tag{3.4}$$

Proof. We begin by proving the first formula. Locally,

$$\begin{aligned}
D_H^\eta(X \cdot \varphi) &= \sum_{j=1}^{2m} b_j \cdot \nabla^\eta_{b_j}(X \cdot \varphi) \\
&= \sum_{j=1}^{2m} b_j \cdot (\nabla^\eta_{b_j} X) \cdot \varphi + b_j \cdot X \cdot (\nabla^\eta_{b_j}\varphi) \\
&= \sum_{j=1}^{2m} b_j \cdot (\nabla^\eta_{b_j} X) \cdot \varphi - X \cdot b_j \cdot (\nabla^\eta_{b_j}\varphi) - 2g(b_j, X)\nabla^\eta_{b_j}\varphi \\
&= -X \cdot D_H^\eta\varphi - 2\nabla_X^\eta\varphi + \sum_{i=1}^{2m} b_j \cdot (\nabla^\eta_{b_j} X) \cdot \varphi.
\end{aligned}$$

Under the identification of the Clifford algebra with the exterior algebra, we have

$$X \cdot Y = X^\flat \wedge Y^\flat - X \lrcorner Y^\flat.$$

Applying this, we obtain

$$\begin{aligned}
D_H^\eta(X \cdot \varphi) &= -X \cdot D_H^\eta\varphi - 2\nabla_X^\eta\varphi + \sum_{j=1}^{2m}((b_j)^\flat \wedge (\nabla^\eta_{b_j} X)^\flat - b_j \lrcorner (\nabla^\eta_{b_j} X)^\flat) \cdot \varphi \\
&= -X \cdot D_H^\eta\varphi - 2\nabla_X^\eta\varphi + \sum_{j=1}^{2m}((b_j)^\flat \wedge \nabla^\eta_{b_j}(X^\flat) - b_j \lrcorner \nabla^\eta_{b_j}(X^\flat) \cdot \varphi.
\end{aligned}$$

Using the local expressions for d_H and d_H^*, we deduce

$$D_H^\eta(X \cdot \varphi) = -X \cdot D_H^\eta\varphi - 2\nabla_X^\eta\varphi + (d_H + d_H^*)X^\flat \cdot \varphi.$$

This proves the first equation. Using it and the formula for $D_H^\eta(f\varphi)$, we obtain

$$(D_H^\eta)^2(f\varphi) = f(D_H^\eta)^2\varphi - 2\nabla^\eta_{\mathrm{grad}_H f}\varphi + (\Delta_H f)\varphi + d_H d_H f \cdot \varphi.$$

Unlike the full differential, d_H does not define a chain complex, i.e. $d_H^2 \neq 0$. More precisely, for $X, Y \in \Gamma(H)$, we have

$$\begin{aligned} d_H d_H f(X,Y) &= X(d_H f(Y)) - Y(d_H f(X)) - d_H f([X,Y]) \\ &= X(df(pr_H(Y))) - Y(df(pr_H(X))) - df(pr_H([X,Y])) \\ &= X(df(Y)) - Y(df(X)) - df([X,Y]) + df(pr_V[X,Y]) \\ &= ddf(X,Y) - d\eta(X,Y)\xi(f). \end{aligned}$$

As $d^2 = 0$, this yields the claim. □

We saw in Section 2.3 that the spinor bundle splits into eigenbundles of the Clifford action of $d\eta$. We would like to know how this splitting behaves under the horizontal Dirac operator. We have already seen that it is stable under any adapted connection, so what remains to see is how it behaves under the Clifford action. We consider the complexified bundle $H_{\mathbb{C}}$ (cf. the beginning of Section 1.2) and the projections

$$\begin{aligned} p_{10}\colon TM &\to H^{(1,0)} & \qquad p_{01}\colon TM &\to H^{(0,1)} \\ X &\mapsto X_H - iJX_H & X &\mapsto X_H + iJX_H, \end{aligned}$$

where X_H is the projection onto H of X. We extend the Clifford action complex-linearly to $H_{\mathbb{C}}$.

Lemma 3.1.6. *Let (M, g, J, η) be a spin metric contact manifold and $\mathbb{S} = \bigoplus \mathbb{S}_{m-2k}$ the splitting (2.21). Let $X \in H$ and $\varphi \in \mathbb{S}_{m-2k}$. Then,*

$$p_{10}(X) \cdot \varphi \in \mathbb{S}_{(m-2(k-1))} \quad \textit{and} \quad p_{01}(X) \cdot \varphi \in \mathbb{S}_{(m-2(k+1))},$$

where, as always $\mathbb{S}_r = 0$ if $r \notin \{-m, -m+2, \ldots, m\}$.

Proof. As the Clifford action is linear, it is enough to show the claim for elements of the basis. Let (e_j, f_j) be an adapted local basis of H. Then, (e_j, f_j, ξ) is a local section of P_U and we can locally trivialise $\mathbb{S}$ by a lift of this section. Using a trivialisation of TM induced by the same section, e_j is represented by s_{2j-1} and f_j by s_{2j}, where (s_j) is the standard basis of $\mathbb{R}^{2m+1}$. Then, we saw in section 2.3 that the spaces $\mathbb{S}_{m-2k}$ are given as

$$\mathbb{S}_{m-2k} = \{\, u(\delta_1) \otimes \cdots \otimes u(\delta_m) \mid \#\{j | \delta_j = -1\} = k \} \,.$$

Recall that $u(\delta) = 2^{-1/2}(1, -\delta i)^T$ and that

$$\begin{aligned} s_{2j-1} \cdot u_{\delta_1,\ldots,\delta_m} &= i(-1)^{j-1} \delta_{m-j+2} \cdots \delta_m u_{\delta_1,\ldots,\delta_{m-j},-\delta_{m-j+1},\delta_{m-j+2},\ldots,\delta_m} \\ s_{2j} \cdot u_{\delta_1,\ldots,\delta_m} &= (-1)^{j-1} \delta_{m-j+1} \cdots \delta_m u_{\delta_1,\ldots,\delta_{m-j},-\delta_{m-j+1},\delta_{m-j+2},\ldots,\delta_m}. \end{aligned}$$

For the Clifford action of $e_j \pm iJe_j = e_j \pm if_j$, we obtain

$$(s_{2j-1} \pm is_{2j}) \cdot u_{\delta_1,\dots,\delta_m} = i(-1)^{j-1}\delta_{m-j+2}\cdots\delta_m(u_{\delta_1,\dots,\delta_{m-j},-\delta_{m-j+1},\delta_{m-j+2},\dots,\delta_m} \\ \pm \delta_{m-j+1}u_{\delta_1,\dots,\delta_{m-j},-\delta_{m-j+1},\delta_{m-j+2},\dots,\delta_m}).$$

If $\pm$ and δ_{m-j+1} have different sign, the right side vanishes. If they have the same sign, we obtain

$$(s_{2j-1} \pm is_{2j}) \cdot u_{\delta_1,\dots,\delta_m} = 2i(-1)^{j-1}\delta_{m-j+2}\cdots\delta_m u_{\delta_1,\dots,\delta_{m-j},-\pm,\delta_{m-j+2},\dots,\delta_m} \\ \in \mathbb{S}_{m-2(k\pm1)}$$

On the other hand, $f_j \pm iJf_j = f_j \mp ie_j = \mp i(e_j \pm if_j)$, so we obtain the same result for $f_j \pm iJf_j$. The result then follows from the definition of p_{10} and p_{01}. □

We can thus split D_H^∇ into two operators that respectively raise and lower the index $(m-2k)$ of the eigenbundle.

Definition. We define the first-order differential operators

$$D_+^\nabla \colon \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(T^*M\otimes\mathbb{S}) \xrightarrow{\simeq} \Gamma(TM\otimes\mathbb{S}) \xrightarrow{p_{10}\otimes id} \Gamma(H^{(1,0)}\otimes\mathbb{S}) \xrightarrow{cl} \Gamma(\mathbb{S}). \quad (3.5)$$

and

$$D_-^\nabla \colon \Gamma(\mathbb{S}) \xrightarrow{\nabla} \Gamma(T^*M\otimes\mathbb{S}) \xrightarrow{\simeq} \Gamma(TM\otimes\mathbb{S}) \xrightarrow{p_{01}\otimes id} \Gamma(H^{(0,1)}\otimes\mathbb{S}) \xrightarrow{cl} \Gamma(\mathbb{S}). \quad (3.6)$$

We write D_+^η and D_-^η for $D_\pm^\nabla$ where $\nabla = \nabla^\eta$. Obviously, $D_H^\nabla = D_+^\nabla + D_-^\nabla$.

These operators are locally given by

$$D_+^\nabla\varphi = \sum_{j=1}^{2m} p_{10}(b_j)\nabla_{b_j}\varphi = \frac{1}{2}\sum_{j=1}^{2m}(b_j - iJb_j)\nabla_{b_j}\varphi,$$
$$D_-^\nabla\varphi = \sum_{j=1}^{2m} p_{01}(b_j)\nabla_{b_j}\varphi = \frac{1}{2}\sum_{j=1}^{2m}(b_j + iJb_j)\nabla_{b_j}\varphi,$$

where (b_j) is a local ON-basis of H. The following proposition is a direct consequence of Lemmas 2.3.2 and 3.1.6.

Proposition 3.1.7. *Let (M,g,J,η) be a spin metric contact manifold and ∇ an adapted connection on M. Then, restricted to the subspaces $\mathbb{S}_{m-2k}$ of the spinor bundle, the operators D_+^∇ and D_-^∇ satisfy*

$$D_+^\nabla(\Gamma(\mathbb{S}_{m-2k})) \subset \Gamma(\mathbb{S}_{m-2(k-1)}) \quad \textit{and} \quad D_-^\nabla(\Gamma(\mathbb{S}_{m-2k})) \subset \Gamma(\mathbb{S}_{m-2(k+1)})$$

Moreover, the two operators are formal adjoints of each other.

Proposition 3.1.8. *Let (M, g, J, η) be a spin metric contact manifold and ∇ an adapted connection on M whose torsion is traceless. Then, the operators D_+^∇ and D_-^∇ are formal adjoints of each other.*

Proof. Let (b_j) a p-synchronous orthonormal basis of H, i.e. $(\nabla b_k)(p) = 0$. As J is parallel under ∇, this implies $(\nabla p_{10}(b_j))(p) = (\nabla p_{01}(b_j))(p) = 0$. Then, at p,

$$
\begin{aligned}
\langle D_+^\nabla \varphi, \psi\rangle &= \sum_{j=1}^{2m} \langle p_{10}(b_j)\nabla_{b_j}\varphi, \psi\rangle \\
&= -\sum_{j=1}^{2m} \langle \nabla_{b_j}\varphi, p_{01}(b_j)\psi\rangle.
\end{aligned}
$$

Note the change from p_{10} to p_{01} which is due to the scalar product being Hermitian, i.e. complex antilinear in the second component. We define $V = V_{\varphi,\psi} \in \Gamma(H_{\mathbb{C}})$ as the unique vector field such that for every $W \in \Gamma(H)$, we have

$$
g(V, W) = \langle \varphi, W \cdot \psi\rangle,
$$

where we extended g complex-bilinearly. Then, arguing as in the proof of Proposition 3.1.1, we find

$$
\langle D_+^\nabla \varphi, \psi\rangle = \langle \varphi, D_-^\nabla \psi\rangle - \sum_{j=1}^{2m} b_j(g(V, p_{01}(b_j))).
$$

Using Lemma 2.2.4 and that (b_j) is synchronous, the second sum becomes

$$
\begin{aligned}
\sum_{j=1}^{2m} b_j(g(V, p_{01}(b_j)) &= \sum_{j=1}^{2m} g(\nabla_{b_j} V, p_{01}(b_j)) - g(V, \nabla_{b_j} p_{01}(b_j)) \\
&= \sum_{j=1}^{2m} g(\nabla_{b_j} V, b_j) + ig(\nabla_{b_j} V, J b_j) \\
&= \sum_{j=1}^{2m} g(\nabla_{b_j} V, b_j) - ig(\nabla_{b_j} JV, b_j).
\end{aligned}
$$

These sums become $\operatorname{div}^\nabla(V) - i \operatorname{div}^\nabla(JV)$ because $g(\nabla_\xi V, \xi) = 0$. As the torsion of ∇ is traceless, Lemma 2.2.4 implies that this is equal to $\operatorname{div}^g(V)$ and $i \operatorname{div}^g(JV)$ respectively. Integrating and using Stokes' theorem yields the claim. ☐

We will come back to these operators in the following section, where we will show that in the case of the Tanaka-Webster connection, $(D_H^\eta)^2$ stabilises the $d\eta$-eigenspaces.

3.2 Weitzenböck and Schrödinger-Lichnerowicz type formulae

Weitzenböck type formulae, which link the square of a Dirac operator with a connection (or Bochner) Laplacian, are a useful tool for vanishing theorems and eigenvalue estimates. In this section, we present such theorems for the horizontal Dirac operators, in particular the one associated with the Tanaka-Webster connection. Such theorems were produced for the Tanaka-Webster connection by Robert Petit in [Pet05]. We give abstract (i.e. coordinate-free) definitions of the necessary connection Laplacians for general adapted connections and a general Weitzenböck-type formula before deducing Petit's formulae for the Tanaka-Webster case.

We begin by defining the horizontal version of the connection Laplacian. Let ∇ be a metric connection on M. Note that while it would geometrically make sense to assume ∇ adapted here, the definition of the horizontal connection Laplacian works for general metric connections and we will need this for the Weitzenböck-type formula later on. We write ∇_H for the restriction of ∇ (as a connection on $\mathbb{S}$) to derivatives in the direction of H (note that the restriction is only in the first argument and thus makes sense for any metric connection). In other words, we consider it as an operator $\nabla_H \colon \Gamma(\mathbb{S}) \to \Gamma(H^* \otimes \mathbb{S})$. If we assume M to be compact, we have an L^2-scalar product on $\mathbb{S}$ that we extend to one on $H^* \otimes \mathbb{S}$ via

$$(\alpha \otimes \varphi, \beta \otimes \psi)_{L^2} = (\alpha, \beta)_{L^2} (\varphi, \psi)_{L^2}.$$

Recall that the scalar product for 1-forms is given pointwise by

$$\langle \alpha, \beta \rangle = \sum_{j,k=1}^{2m} \alpha(b_j)\beta(b_k)\langle b_j, b_k \rangle,$$

where (b_j) is some basis of H_p (a basis for H suffices as the forms are in H^*). Using the L^2-product, we can consider the formal adjoint operator

$$(\nabla_H)^* \colon \Gamma(H^* \otimes \mathbb{S}) \longrightarrow \Gamma(\mathbb{S}).$$

Definition. Let (M, g, J, η) be a spin metric contact manifold and ∇ a metric connection on M. The *horizontal connection Laplacian* is the operator

$$\Delta_H^{\nabla} = (\nabla_H)^* \circ \nabla_H \colon \Gamma(\mathbb{S}) \longrightarrow \Gamma(\mathbb{S}).$$

We write Δ_H^{η} for the connection Laplacian of the Tanaka-Webster connection.

Remark. The (horizontal) connection Laplacians are not specific to spinor bundles but can be formed on any euclidean or Hermitian vector bundle with a bundle connection.

We want to have a local formula for this operator. As a preparation, we remind the reader of the metric duals. For $v \in T_pM$, we write $v^\flat$ for the one-form given by $v^\flat(w) = g_p(w, v)$. Conversely, for $\alpha \in T_pM^*$, we write $\alpha^\sharp$ for the vector that satisfies $g_p(w, \alpha^\sharp) = \alpha(w)$ for all $w \in T_pM$, i.e. the vector that satisfies $(\alpha^\sharp)^\flat = \alpha$. Clearly, for $v \in H$, $v^\flat \in H^*$ and vice versa. These definitions extend to vector fields and differential forms in the obvious way.

Lemma 3.2.1. *Let (M, g, J, η) be a closed (i.e. compact and without boundary) spin metric contact manifold, ∇ a metric connection on M and (b_j) a local ON-basis of H. Then, locally, the horizontal connection Laplacian is given by*

$$\Delta_H^\nabla \varphi = \sum_{j=1}^{2m} -\nabla_{b_j}\nabla_{b_j}\varphi - \operatorname{div}^g(b_j)\nabla_{b_j}\varphi. \tag{3.7}$$

Proof. The arguments are the same as for the well-known local formula for the full connection Laplacian. We begin by establishing a local formula for ∇_H^*. Let $\alpha \in \Gamma(H^*)$, $\varphi, \psi \in \Gamma(\mathbb{S})$ and let (b^j) be the dual basis of (b_j). Locally, we have

$$\begin{aligned}\langle \nabla_H \varphi, \alpha \otimes \psi\rangle &= \sum_{j=1}^{2m} \langle b^j \otimes \nabla_{b_j}\varphi, \alpha \otimes \psi\rangle \\ &= \sum_{j=1}^{2m} \alpha(b_j)\left(b_j(\langle \varphi, \psi\rangle) - \langle \varphi, \nabla_{b_j}\psi\rangle\right) \\ &= \alpha^\sharp(\langle \varphi, \psi\rangle) - \langle \varphi, \nabla_{\alpha^\sharp}\psi\rangle.\end{aligned}$$

Recall the following product rule for the divergence:

$$\operatorname{div}^g(fX) = X(f) + f\operatorname{div}^g(X).$$

Applying this, we obtain

$$\langle \nabla_H \varphi, \alpha \otimes \psi\rangle = \operatorname{div}^g(\langle \varphi, \psi\rangle \alpha^\sharp) - \langle \varphi, \psi\rangle \operatorname{div}^g(\alpha^\sharp) - \langle \varphi, \nabla_{\alpha^\sharp}\psi\rangle.$$

Integrating over M and using Stokes' Theorem, we obtain

$$(\nabla_H \varphi, \alpha \otimes \psi)_{L^2} = (\varphi, -\nabla_{\alpha^\sharp}\psi - \operatorname{div}(\alpha^\sharp)\psi)_{L^2}.$$

As this holds for any $\varphi \in \Gamma(\mathbb{S})$, we obtain

$$\nabla_H^*(\alpha \otimes \psi) = -\nabla_{\alpha^\sharp}\psi - \operatorname{div}^g(\alpha^\sharp)\psi \tag{3.8}$$

for any $\alpha \in \Gamma(H^*)$ and $\psi \in \Gamma(\mathbb{S})$.

Next, as $(b_j)^\flat = b^j$, we obtain the local formula

$$\begin{aligned}\nabla_H^* \nabla_H \varphi &= \sum_{j=1}^{2m} \nabla_H^* \left(b^j \otimes \nabla_{b_j} \varphi \right) \\ &= \sum_{j=1}^{2m} -\nabla_{b_j} \nabla_{b_j} \varphi - \operatorname{div}^g(b_j) \nabla_{b_j} \varphi.\end{aligned}$$

This yields the claim. □

We quickly prove a technical result that we will use later. It is probably well-known but we did not find a reference.

Lemma 3.2.2. *Let (M^n, g) be spin and $S\colon TM \to TM$ a symmetric endomorphism. Let further (b_j) be a local orthonormal basis of TM. Then, we have for any $\varphi \in \Gamma(\mathbb{S})$*

$$\sum_{j=1}^{n} b_j \cdot S(b_j) \cdot \varphi = \sum_{j=1}^{n} S(b_j) \cdot b_j \cdot \varphi = -\operatorname{tr} S \cdot \varphi. \tag{3.9}$$

In the case of a metric contact manifold, the same result holds for an orthonormal basis and a symmetric endomorphism of H, where the trace will be of the operator on H.

Proof. We have $Sb_j = \sum_k g(Sb_j, b_k) b_k$. Thus, we obtain

$$\sum_{j=1}^{n} b_j \cdot S(b_j) = \sum_{j,k} g(S(b_j), b_k) b_j \cdot b_k = \sum_{j,k} g(b_j, S(b_k)) b_j \cdot b_k = \sum_{k=1}^{n} S(b_k) \cdot b_k. \tag{3.10}$$

On the other hand, from standard Clifford multiplication, we deduce that

$$\begin{aligned}\sum_{j=1}^{n} b_j \cdot S(b_j) &= -\sum_{j=1}^{n} S(b_j) \cdot b_j + 2g(S(b_j), b_j) \\ &\overset{(3.10)}{=} -\sum_{j=1}^{n} b_j \cdot S(b_j) + 2g(S(b_j), b_j).\end{aligned}$$

□

As a first step towards a Weitzenböck-type formula, we give a straightforward comparison formula for $(D_H^\nabla)^2$ and Δ_H^∇. This comparison will involve the following first-order differential operator: Denote the restriction of the torsion tensor (considered as a (3,0)-tensor) to H in all three arguments as $T_H\colon H \times H \times H \to \mathbb{R}$ and let it induce a mapping $c(T_H)\colon \Gamma(H^* \otimes \mathbb{S}) \to \Gamma(\mathbb{S})$ via $c(T_H)(\alpha \otimes \varphi) = T_H(\alpha^\sharp; \cdot, \cdot) \cdot \varphi$. We then define the first-order differential operator

$$D^\nabla_{T,H}\colon \Gamma(\mathbb{S}) \xrightarrow{\nabla_H} \Gamma(H^* \otimes \mathbb{S}) \xrightarrow{c(T_H)} \Gamma(\mathbb{S}).$$

In a local orthonormal basis (b_j) of H, this operator is given by

$$D^\nabla_{T,H}\varphi = \sum_{l=1}^{2m} \sum_{1\le j<k\le 2m} T(b_l; b_j, b_k) b_j \cdot b_k \cdot \nabla_{b_l}\varphi.$$

We are now ready to prove the first result of this section, a direct comparison between the square of D^∇_H and the connection Laplacian.

Lemma 3.2.3. *Let (M, g, η, J) be a closed spin metric contact manifold and ∇ an adapted connection with traceless torsion, i.e. whose horizontal Dirac operator D^∇_H is formally self-adjoint. Then, we have the following relation between its square $(D^\nabla_H)^2$ and the connection Laplacian:*

$$(D^\nabla_H)^2 = \nabla^*_H \nabla_H - d\eta \cdot \nabla_\xi - D^\nabla_{T,H} + K^\nabla, \tag{3.11}$$

where $K^\nabla\colon \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is the endomorphism given locally by

$$K^\nabla\varphi = \frac{1}{8} \sum_{j,k,\alpha,\beta=1}^{2m} R^\nabla(b_j, b_k, b_\alpha, b_\beta) b_j \cdot b_k \cdot b_\alpha \cdot b_\beta \cdot \varphi.$$

Proof. Let (b_j) be a local orthonormal basis of H that is p-synchronous with respect to ∇, i.e. $\nabla_{b_j} b_k(p) = 0$. Then, at p, we have

$$\begin{aligned}
(D^\nabla_H)^2\varphi &= \left(\sum_{j=1}^{2m} b_j \cdot \nabla_{b_j}\right)\left(\sum_{k=1}^{2m} b_k \cdot \nabla_{b_k}\right)\varphi \\
&= \sum_{j=1}^{2m} b_j^2 \nabla_{b_j}\nabla_{b_j}\varphi + \sum_{j<k} b_j b_k (\nabla_{b_j}\nabla_{b_k} - \nabla_{b_k}\nabla_{b_j})\varphi \\
&= -\sum_{j=1}^{2m} \nabla_{b_j}\nabla_{b_j}\varphi + \sum_{j<k} b_j b_k (R^{\mathbb{S}}(b_j, b_k) - \nabla_{T(b_j,b_k)})\varphi \\
&= -\sum_{j=1}^{2m} \nabla_{b_j}\nabla_{b_j}\varphi + \sum_{j<k} b_j b_k \Bigg(R^{\mathbb{S}}(b_j, b_k) - T(\xi; b_j, b_k)\nabla_\xi \\
&\qquad\qquad - \sum_{l=1}^{2m} T(b_l; b_j, b_k)\nabla_{b_l}\Bigg)\varphi,
\end{aligned}$$

where $R^{\mathbb{S}}$ is the curvature endomorphism of ∇ on $\mathbb{S}$. As we assumed $\operatorname{tr} T = 0$, we know that $\operatorname{div}^g(b_j) = \operatorname{div}^\nabla(b_j) = 0$ as b_j is p-synchronous with respect to ∇, and therefore, the first summand is equal to the connection Laplacian. By Theorem 1.4.3, $T(\xi,\cdot,\cdot) = d\eta$ and the last summand is exactly $D^\nabla_{T,H}$. Finally, by Lemma 2.2.2, we have

$$R^{\mathbb{S}}(b_j,b_k)\varphi = \frac{1}{4}\sum_{\alpha,\beta=1}^{2m} R^\nabla(b_j,b_k,b_\alpha,b_\beta)b_\alpha\cdot b_\beta\cdot\varphi.$$

This yields the claimed form of K^∇. The endomorphism K^∇ is independent of the basis chosen because all other terms in the formula are. $\square$

Remark. Note that while D^∇_H derives only in the direction of H, its square involves a derivative in the direction of ξ. This is a typical behaviour of such horizontal operators and the derivative in the ξ-direction is actually crucial for the analytic properties of the second-order operator $(D^\nabla_H)^2$. We will discuss this in detail in Chapter 4.

While the first-order derivative in the direction of ξ is unavoidable in this formula, we can get rid of the first-order operator $D^\nabla_{T,H}$ by using the connection Laplacian of an appropriate connection. Doing this, one obtains the following Weitzenböck-type formula.

Theorem 3.2.4 (Weitzenböck-type formula). *Let (M,g,η,J) be a closed spin metric contact manifold and ∇ an adapted connection with traceless torsion T, i.e. whose horizontal Dirac operator D^∇_H is formally self-adjoint. Then, there exists a connection ∇^W on $\Gamma(\mathbb{S})$ and an endomorphism E of $\mathbb{S}$ such that*

$$(D^\nabla_H)^2 = (\nabla^W_H)^*\circ\nabla^W_H - d\eta\cdot\nabla_\xi + E. \tag{3.12}$$

The endomorphism E is given by

$$E\varphi = \tfrac{1}{2}\operatorname{tr}_H(\nabla T_H)\cdot\varphi + K^\nabla\varphi + \frac{1}{4}\sum_{j=1}^{2m} T_H(b_j;\cdot,\cdot)\cdot T_H(b_j;\cdot,\cdot)\cdot\varphi,$$

where T_H is the restriction of T to H in all three arguments and the trace of ∇T_H is taken in ∇ and the first argument of T_H (as a (3,0)-tensor), i.e. $\operatorname{tr}_H(\nabla T_H) = \sum_j(\nabla_{b_j}T_H)(b_j;\cdot,\cdot)$. The connection ∇^W is given by

$$\begin{aligned}\nabla^W_X\varphi &= \nabla_X\varphi + \tfrac{1}{2}T_H(X;\cdot,\cdot)\cdot\varphi \qquad\qquad \textit{for } X\in H,\\ \nabla^W_\xi\varphi &= \nabla_\xi\varphi,\end{aligned}$$

and induced by the connection ∇^W on TM defined by

$$g(\nabla^W_X Y, Z) = g(\nabla_X Y, Z) + T(X;Y,Z) \qquad g(\nabla^W_X Y, \xi) = 0$$
$$\nabla^W_\xi X = \nabla_\xi X \qquad \nabla^W \xi = 0$$

for any $X, Y, Z \in \Gamma(H)$. If, in the notation of Theorem 1.4.3, $\nabla = \nabla(\omega, B, \Phi)$, then ∇^W is adapted if and only if the manifold is CR and $\omega = 0$. Then, $\nabla^W = \nabla(0, \frac{1}{2}B, \Phi)$.

Proof. Let (b_j) a p-synchronous local basis with respect to ∇. We write $T_X = T_H(X;\cdot,\cdot)$. Then, defining ∇^W on $\mathbb{S}$ as above, the local formula for horizontal connection Laplacians yields (using that $\operatorname{div}^g(b_j) = \operatorname{div}^\nabla(b_j) = 0$)

$$\begin{aligned}
(\nabla^W_H)^*(\nabla^W_H) &= -\sum_{j=1}^{2m} (\nabla_{b_j} + \tfrac{1}{2}T_{b_j})(\nabla_{b_j} + \tfrac{1}{2}T_{b_j}) \\
&= -\sum_{j=1}^{2m} \nabla_{b_j}\nabla_{b_j} + \tfrac{1}{2}\nabla_{b_j}T_{b_j} + \tfrac{1}{2}T_{b_j}\nabla_{b_j} + \tfrac{1}{4}T_{b_j}\cdot T_{b_j} \\
&= \Delta^\nabla_H - \sum_{j=1}^{2m} T_{b_j}\nabla_{b_j} + \tfrac{1}{2}\nabla_{b_j}(T_{b_j}) + \tfrac{1}{4}T_{b_j}\cdot T_{b_j}.
\end{aligned}$$

Now, $\sum_j T_{b_j}\nabla_{b_j}$ is exactly $D^\nabla_{T,H}$. Thus, comparing the above formula with (3.11), we deduce

$$(D^\nabla_H)^2 = (\nabla^W_H)^*\nabla^W_H - d\eta\cdot\nabla_\xi + \sum_{j=1}^{2m} \tfrac{1}{2}\nabla_{b_j}(T_{b_j}) + \tfrac{1}{4}T_{b_j}\cdot T_{b_j}\cdot + K^\nabla.$$

As (b_j) is p-synchronous, $\nabla_{b_j}(T_{b_j}) = (\nabla_{b_j}T)(b_j,\cdot,\cdot)$. This yields the claimed formula. The derivative ∇T_H is a (4,0)-tensor and its trace is therefore well-defined and thus, $\sum_j T_{b_j}T_{b_j}$ is well defined (i.e. independent of the basis) because all other terms in the formula are.

We now consider ∇^W as described on TM and show that it does induce the connection on $\mathbb{S}$ define above. As T is skew-symmetric in the last two arguments, it is clear that ∇^W as it is defined on TM is metric. We write T^W, A^W for its torsion and potential and A for the potential of ∇. By the definition of ∇^W, $A^W - A = \frac{1}{2}T_H$. Then, from the formulae for the relationship between torsion and potential (1.29), we deduce

$$T^W = -A^W + 3\mathfrak{b}A^W = -A + 3\mathfrak{b}A - \tfrac{1}{2}T_H + 3\mathfrak{b}\tfrac{1}{2}T_H = T - \tfrac{1}{2}(T_H - 3\mathfrak{b}T_H). \quad (3.13)$$

Thus, by (2.16), for the spinor connection we have

$$\begin{aligned}\nabla^W_X\varphi &= \nabla^g_X\varphi - T^W(X;\cdot,\cdot)\cdot\varphi + \tfrac{3}{2}\mathfrak{b}T^W(X,\cdot,\cdot)\cdot\varphi\\ &= \nabla^g_X\varphi - (T - \tfrac{1}{2}(T_H - 3\mathfrak{b}T_H))(X;\cdot,\cdot)\cdot\varphi + \tfrac{3}{2}\mathfrak{b}(T - \tfrac{1}{2}(T_H - 3\mathfrak{b}T_H))(X,\cdot,\cdot)\cdot\varphi\\ &= \nabla_X\varphi + \tfrac{1}{2}T_X\cdot\varphi\end{aligned}$$

for any $X \in \Gamma(H)$, i.e. T^W as defined on TM does induce T^W as defined on $\mathbb{S}$.

Now let ∇ be defined as $\nabla(\omega, B, \Phi)$. Then, we use (1.30) to see that the full torsion tensor is given by

$$T = N^{0,2} + \tfrac{9}{8}\omega - \tfrac{3}{8}\mathfrak{M}\omega + B + \xi\otimes d\eta - \tfrac{1}{2}\eta\wedge(J\mathcal{J}) + \eta\wedge\Phi.$$

As T^W differs from T only on H we only consider these parts. Recalling formula (1.44) for $\mathfrak{b}T$ and using (3.13), we obtain

$$T^W_H = \tfrac{1}{2}T_H + \tfrac{3}{2}\mathfrak{b}T_H = \tfrac{1}{2}(N^{0,2} + \tfrac{9}{8}\omega - \tfrac{3}{8}\mathfrak{M}\omega + B) + \tfrac{3}{2}\omega = \tfrac{1}{2}N^{0,2} + \tfrac{33}{16}\omega - \tfrac{3}{16}\mathfrak{M}\omega + \tfrac{1}{2}B.$$

For T^W to be adapted we would need ω^W and B^W such that

$$T^W_H = N^{0,2} + \tfrac{9}{8}\omega^W - \tfrac{3}{8}\mathfrak{M}\omega^W + B^W.$$

Clearly, this implies $N^{0,2} = 0$ and the manifold must thus be CR. By Corollary 1.3.3, $\mathfrak{M}\omega$ is not totally skew-symmetric and therefore we obtain the contradictory conditions $\omega^W = \tfrac{33}{18}\omega$ and $\omega^W = \tfrac{1}{2}\omega$. Thus, ∇^W can only be adapted if ω vanishes. In that case, choosing $B^W = \tfrac{1}{2}B$ will yield the required form. $\square$

Remark. Note that in the above theorem, ∇^W_ξ does not enter into Δ^{∇^W} and can thus be freely chosen (within the limits set by the requirement that ∇^W be a metric connection). We made a choice here in such a way that made it most likely for ∇^W to be adapted again.

One can now deduce Schrödinger-Lichnerowicz type theorems by calculating the curvature term K^∇ and making $D_{T,H}$ more explicit. We will only do this for the Tanaka-Webster connection on CR manifolds as this is the operator we are mainly interested in. Formulae for other adapted connections, including the generalised Tanaka-Webster connection on metric contact manifolds, will involve additional torsion terms.

Theorem 3.2.5 (Schrödinger-Lichnerowicz formula for D^η_H, [Pet05, Prop 4.2])**.** *Let (M, H, J, η) be a closed spin strictly pseudoconvex CR manifold and ∇^η its Tanaka-Webster connection. Then, we have the following relation between the square $(D^\eta_H)^2$ of the Tanaka-Webster operator and the connection Laplacian:*

$$(D^\eta_H)^2 = (\nabla^\eta_H)^*\nabla^\eta_H - d\eta\cdot\nabla^\eta_\xi + \frac{1}{4}\,\mathrm{scal}^\eta\,. \tag{3.14}$$

Proof. Starting with the Weitzenböck-type formula, we deduce that $\nabla^W = \nabla^\eta$ because $T^\eta_H = 0$. Thus, we only need to prove that the endomorphism E is given by multiplication with $\frac{1}{4}\operatorname{scal}^\eta$. Again because $T^\eta_H = 0$, E reduces to K^η. In what follows, when we write sums over indices like $j < k$ we always mean sum over all indices from 1 to $2m$ with the additional restriction noted. Let (b_j) be a p-synchronous (with respect to ∇^η) orthonormal basis. Then, at p, we have

$$\begin{aligned}
K^\nabla &= \frac{1}{4}\sum_{j<k}\sum_{\alpha,\beta=1}^{2m} R^\eta(b_j,b_k,b_\alpha,b_\beta)b_jb_kb_\alpha b_\beta \\
&= \frac{1}{8}\sum_{j,k,\alpha,\beta=1}^{2m} R^\eta(b_j,b_k,b_\alpha,b_\beta)b_jb_kb_\alpha b_\beta \\
&= \frac{1}{8}\sum_{\beta=1}^{2m}\sum_{j\neq k\neq\alpha} R^\eta(b_j,b_k,b_\alpha,b_\beta)b_jb_kb_\alpha b_\beta + \frac{1}{4}\sum_{\beta=1}^{2m}\sum_{j\neq k} R^\eta(b_j,b_k,b_j,b_\beta)b_jb_kb_jb_\beta.
\end{aligned} \tag{3.15}$$

We consider the first sum. As $b_jb_kb_\alpha b_\beta = b_kb_\alpha b_jb_\beta = b_\alpha b_jb_kb_\beta$, by a renaming of indices it is equal to

$$\begin{aligned}
&\frac{1}{24}\sum_{\beta=1}^{2m}\sum_{j\neq k\neq\alpha} \left(R^\eta(b_j,b_k,b_\alpha,b_\beta) + R^\eta(b_k,b_\alpha,b_j,b_\beta) + R^\eta(b_\alpha,b_j,b_k,b_\beta)\right) b_jb_kb_\alpha b_\beta \\
\overset{(1.45)}{=}&\frac{1}{24}\sum_{\beta=1}^{2m}\sum_{j\neq k\neq\alpha} (d\eta(b_j,b_k)g(\tau b_\alpha,b_\beta) + d\eta(b_k,b_\alpha)g(\tau b_j,b_\beta) \\
&\qquad + d\eta(b_\alpha,b_j)g(\tau b_k,b_\beta))\, b_jb_kb_\alpha b_\beta \\
=&\frac{1}{8}\sum_{\beta=1}^{2m}\sum_{j\neq k\neq\alpha} d\eta(b_j,b_k)g(\tau b_\alpha,b_\beta)b_jb_kb_\alpha b_\beta \\
=&\frac{1}{8}\sum_{j,k,\alpha,\beta=1}^{2m} d\eta(b_j,b_k)g(\tau b_\alpha,b_\beta)b_jb_kb_\alpha b_\beta - \frac{1}{4}\sum_{j,k,\beta=1}^{2m} d\eta(b_j,b_k)g(\tau b_j,b_\beta)b_jb_kb_jb_\beta.
\end{aligned}$$

Using that $v = \sum_{j=1}^{2m} g(v,b_j)b_j$ for any $v \in H$ and $d\eta = 2g(J\cdot,\cdot)$, these sums transform to

$$\frac{1}{8}d\eta\sum_{\alpha=1}^{2m} b_\alpha\tau b_\alpha + \frac{1}{2}\sum_{\beta=1}^{2m}(\tau Jb_\beta)b_\beta.$$

We know that τ is symmetric and traceless (Lemma 1.4.4). Furthermore,

$$\tau J = -\frac{1}{2}J\mathcal{J}J = \frac{1}{2}J^2\mathcal{J} = -\frac{1}{2}\mathcal{J}$$

is also symmetric and traceless (Proposition 1.1.6) and we therefore deduce from (3.9) that both sums are zero. We are left to consider the second sum in (3.15). As $R^\eta(\cdot,\cdot,\xi,\cdot)=0$, we see that

$$\begin{aligned}
&\sum_{\beta=1}^{2m}\sum_{j\neq k} R^\eta(b_j,b_k,b_j,b_\beta)b_jb_kb_jb_\beta\\
&=-\sum_{\beta=1}^{2m}\sum_{k=1}^{2m}\mathrm{Ric}^\eta(b_k,b_\beta)b_kb_\beta\\
&=-\sum_{k=1}^{2m} b_k\,\mathrm{Ric}^\eta(b_k),
\end{aligned}$$

where we understand Ric^η as an endomorphism in the usual way, i.e. defined via $g(\mathrm{Ric}^\eta(X),Y)=\mathrm{Ric}^\eta(X,Y)$ for any $X,Y\in TM$. As $R^\eta(\cdot,\cdot,\cdot,\xi)=0$, we have $\mathrm{Ric}^\eta(\cdot,\xi)=0$ and thus, in the last step we do obtain $\mathrm{Ric}^\eta(b_k)$ instead of its projection to H. As Ric^η is symmetric on H by (1.47), we deduce from (3.9) that the sum is equal to the trace of Ric^η. This yields $K^{\nabla^\eta}=\frac{1}{4}\,\mathrm{scal}^\eta$. □

Apart from its usual applications, the Schrödinger-Lichernowicz type formula has an additional use in the context of CR geometry: We can use it to determine the relationship between the horizontal Dirac operator and the splitting of $\mathbb{S}$ into eigenspaces of $d\eta$. To determine this relationship we consider the operators

$$D^\eta_\pm\colon \Gamma(\mathbb{S}_{m-2k})\to\Gamma(\mathbb{S}_{m-2(k\mp1)})$$

introduced in (3.5).

Corollary 3.2.6. *Let (M,H,J,η) be a closed spin strictly pseudoconvex CR manifold and ∇^η its Tanaka-Webster connection. Then, we have the following identities for $D^\eta_\pm$:*

$$(D^\eta_+)^2=0,\quad (D^\eta_-)^2=0\quad\textit{and}\quad (D^\eta_H)^2=D^\eta_+D^\eta_-+D^\eta_-D^\eta_+.$$

In particular, $(D^\eta_H)^2$ stabilises the $d\eta$-eigenspaces, i.e.

$$(D^\eta_H)^2\big(\Gamma(\mathbb{S}_{m-2k})\big)\subset\Gamma(\mathbb{S}_{m-2k}).$$

Proof. We know that the eigenspaces are stable under derivatives. Thus, we derive from the local formula in Lemma 3.2.1 that they are stable under the connection Laplacian. Furthermore, they are trivially stable under the Clifford action of $d\eta$ and multiplication with functions. Then, the Schrödinger-Lichnerowicz type formula implies that the eigenspaces are stable under $(D^\eta_H)^2$.

Then, because $(D^\eta_+)^2$ and $(D^\eta_-)^2$ map $\Gamma(\mathbb{S}_{m-2k})$ to $\Gamma(\mathbb{S}_{m-2(k\mp2)})$, they must be zero and $(D^\eta_H)^2$ must be equal to $D^\eta_+D^\eta_-+D^\eta_-D^\eta_+$. □

The downside to the Schrödinger-Lichnerowicz type formula we have proven here is that it still involves a first-order derivative (in the direction of ξ). As we will see later (see (4.38) and remark thereafter), this cannot be alleviated by choosing a suitable Weitzenböck connection instead of ∇^η for the connection Laplacian. We can get rid of this derivative and obtain the classical form "connection Laplacian plus zeroth-order terms" if we use a different kind of Laplacian. Recall that for a CR manifold, the complexified horizontal bundle splits into the $\pm i$-eigenspaces of J as

$$H_{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)}, \quad H^{(1,0)} = E(J,i),\ H^{(0,1)} = \overline{H^{(1,0)}} = E(J,-i),$$

where $E(J,\lambda)$ denotes the eigenspaces of J associated with the eigenvalue λ. We extend the scalar product to $H_{\mathbb{C}}$ via $g_p(zX, wY) = z\overline{w}g_p(X,Y)$ for any $z,w \in \mathbb{C}$ and $X,Y \in H_p$. Given an adapted orthonormal basis (e_j, f_j) of H with dual (e^j, f^j), we obtain an orthonormal basis of $H^{(1,0)}$ from

$$z_j = \tfrac{1}{\sqrt{2}}(e_j - if_j).$$

The complex conjugates $(\overline{z_j})$ form a basis of $H^{(0,1)}$. The dual basis is given by

$$z^j = \tfrac{1}{\sqrt{2}}(e^j + if^j).$$

Using the metric, one again defines $v^\flat$ via $v^\flat(w) = g(w,v)$ for any $v \in H_{\mathbb{C}}$. Then, $z_j^\flat = z^j$ and thus, $\cdot^\flat$ maps $H^{(1,0)}$ to $(H^{(1,0)})^*$ and $H^{(0,1)}$ to $(H^{(0,1)})^*$. Note that this map is complex-antilinear, i.e. $(zv)^\flat = \overline{z}v^\flat$ for any $z \in C$ and $v \in H_{\mathbb{C}}$. The inverse map $\cdot^\sharp$ is extended analogously and with analogous properties.

Any adapted connection ∇ extends complex-linearly to elements of $H_{\mathbb{C}}$ in both arguments. Note that the derivative is not metric in the usual (real) sense anymore, but must take into account the antilinearity in the second argument of g and the Hermitian product $\langle\cdot,\cdot\rangle$ on $\mathbb{S}$:

$$Z(g(V,W)) = g(\nabla_Z V, W) + g(V, \nabla_{\overline{Z}} W) \quad \text{and} \quad Z(\langle\varphi,\psi\rangle) = \langle\nabla_Z\varphi, W\rangle + \langle\varphi, \nabla_{\overline{Z}}\psi\rangle$$

for any $V,W,Z \in \Gamma(H_{\mathbb{C}})$ and $\varphi,\psi \in \Gamma(\mathbb{S})$. In this sense, ∇ can be interpreted as a mapping $\nabla\colon \Gamma(\mathbb{S}) \to \Gamma(H_{\mathbb{C}}^*{\otimes}\mathbb{S})$ and we can restrict it to the subspaces $(H^{(1,0)})^* \subset H_{\mathbb{C}}^*$ and $(H^{(0,1)})^* \subset H_{\mathbb{C}}^*$:

$$\nabla_{(1,0)}\colon \Gamma(\mathbb{S}) \longrightarrow \Gamma((H^{(1,0)})^* \otimes \mathbb{S}), \quad \nabla_{(0,1)}\colon \Gamma(\mathbb{S}) \longrightarrow \Gamma((H^{(0,1)})^* \otimes \mathbb{S}),$$

i.e.

$$(\nabla_{(1,0)})_Z\varphi = \nabla_Z\varphi = \nabla_X\varphi - i\nabla_{JX}\varphi, \quad (\nabla_{(0,1)})_{\overline{Z}}\varphi = \nabla_{\overline{Z}}\varphi = \nabla_X\varphi + i\nabla_{JX}\varphi,$$

where $Z = X - iJX \in H^{(1,0)}$. Then, we can consider the adjoints

$$(\nabla_{(1,0)})^*\colon \Gamma((H^{(1,0)})^* \otimes \mathbb{S}) \longrightarrow \Gamma(\mathbb{S}), \quad (\nabla_{(0,1)})^*\colon \Gamma((H^{(0,1)})^* \otimes \mathbb{S}) \longrightarrow \Gamma(\mathbb{S})$$

and the *CR connection Laplacians*

$$\Delta^{\nabla}_{(1,0)} = (\nabla_{(1,0)})^* \circ \nabla_{(1,0)} \colon \Gamma(\mathbb{S}) \longrightarrow \Gamma(\mathbb{S}), \quad \Delta^{\nabla}_{(0,1)} = (\nabla_{(0,1)})^* \circ \nabla_{(0,1)} \colon \Gamma(\mathbb{S}) \longrightarrow \Gamma(\mathbb{S}).$$

We will write $\Delta^{\eta}_{(1,0)}$ and $\Delta^{\eta}_{(0,1)}$ for the CR connection Laplacians associated with ∇^{η}.

In order to compare these Laplacians with the square of the horizontal Dirac operator, we will need a local formula again.

Lemma 3.2.7. *Let (M, H, J, η) be a closed spin strictly pseudoconvex CR manifold, ∇ an adapted connection on M and (b_j) a local ON-basis of H. Then, locally, the CR connection Laplacians are given by*

$$\Delta^{\nabla}_{(1,0)}\varphi = -\sum_{j=1}^{m} \nabla_{\overline{z_j}} \nabla_{z_j}\varphi + \operatorname{div}^g(\overline{z_j}),$$
$$\Delta^{\nabla}_{(0,1)}\varphi = -\sum_{j=1}^{m} \nabla_{z_j} \nabla_{\overline{z_j}}\varphi + \operatorname{div}^g(z_j),$$

where $z_j = e_j - i f_j$ and (e_j, f_j) is a local adapted orthonormal basis of H and div *is extended $\mathbb{C}$-linearly.*

Proof. We will only prove the formula for $\Delta^{\nabla}_{(1,0)}$, the proof for $\Delta^{\nabla}_{(0,1)}$ is completely analogous. The connection $\nabla_{(1,0)}$ is locally given by

$$\nabla_{(1,0)}\varphi = \sum_{j=1}^{m} z^j \otimes \nabla_{z_j}\varphi.$$

We begin by providing a local formula for $(\nabla_{(1,0)})^*$. Let $\alpha = \sum \alpha_j z^j \in \Gamma((H^{(1,0)})^*)$ Locally, we have that

$$\begin{aligned}
\langle \nabla_{(1,0)}\varphi, \alpha \otimes \psi \rangle &= \sum_{j=1}^{m} \langle z^j \otimes \nabla_{z_j}\varphi, \alpha \otimes \psi \rangle \\
&= \sum_{j=1}^{2m} g(z^j, \alpha) \langle \nabla_{z_j}\varphi, \psi \rangle \\
&= \sum_{j=1}^{2m} \overline{\alpha_j} \left(z_j(\langle \varphi, \psi \rangle) - \langle \varphi, \nabla_{\overline{z_j}}\psi \rangle \right) \\
&= \alpha^\sharp(\langle \varphi, \psi \rangle) - \langle \varphi, \nabla_{\overline{\alpha^\sharp}}\psi \rangle \\
&= \operatorname{div}^g(\langle \varphi, \psi \rangle \cdot \alpha^\sharp) - \operatorname{div}^g(\alpha^\sharp)\langle \varphi, \psi \rangle - \langle \varphi, \nabla_{\overline{\alpha^\sharp}}\psi \rangle.
\end{aligned}$$

As before, integrating yields

$$(\nabla_{(1,0)})^*(\alpha \otimes \psi) = -\operatorname{div}^g(\overline{\alpha}^\sharp) - \nabla_{\overline{\alpha^\sharp}}\psi.$$

As $(z^j)^\sharp = z_j$, combining the local expressions for $\nabla_{(1,0)}$ and $(\nabla_{(1,0)})^*$ then yields the claim. □

We are now ready to prove a new Schrödinger-Lichnerowicz type formula involving the CR connection Laplacians.

Theorem 3.2.8 (CR Schrödinger-Lichnerowicz type formula, [Pet05, Prop 4.2])**.** *Let (M, H, J, η) be a closed spin strictly pseudoconvex CR manifold and ∇^η its Tanaka-Webster connection. Then, we have the following relation between square $(D^\eta_H)^2$ of the Tanaka-Webster operator and the CR connection Laplacians:*

$$(D^\eta_H)^2 = (1 - \tfrac{i}{2m} d\eta) \cdot \Delta^\eta_{(1,0)} + (1 + \tfrac{i}{2m} d\eta) \cdot \Delta^\eta_{(0,1)} + \tfrac{1}{4}\operatorname{scal}^\eta - \tfrac{1}{2m} d\eta \cdot \operatorname{Ric}^J \cdot, \quad (3.16)$$

where Ric^J *is the two-form defined by*

$$\operatorname{Ric}^J(X, Y) = \operatorname{Ric}^\eta_+(X, JY) = \tfrac{1}{2}\left(\operatorname{Ric}^\eta(X, JY) + \operatorname{Ric}^\eta(JX, J^2Y)\right).$$

Proof. Let (e_j, f_j) be a p-synchronous (with respect to ∇^η) adapted basis and $z_j = \frac{1}{\sqrt{2}}(e_j - if_j)$. Then, $\operatorname{div}^g(\overline{z_j}) = \operatorname{div}^{\nabla^\eta}(\overline{z_j}) = 0$, and from the local formula for the first CR connection Laplacian, at p we obtain

$$\begin{aligned}
2\Delta^{(1,0)} &= -\sum_{j=1}^m \nabla^\eta_{e_j+if_j}\nabla^\eta_{e_j-if_j} \\
&= -\sum_{j=1}^m \nabla^\eta_{e_j}\nabla^\eta_{e_j} + \nabla^\eta_{f_j}\nabla^\eta_{f_j} + i(\nabla^\eta_{f_j}\nabla^\eta_{e_j} - \nabla^\eta_{e_j}\nabla^\eta_{f_j}) \\
&= \Delta^\eta_H - i2m\nabla^\eta_\xi + i\sum_{j=1}^m R(e_j, f_j),
\end{aligned}$$

where we used that $[e_j, f_j] = -T(e_j, f_j)$ because the basis is synchronous. Here and in what follows, R is the curvature endomorphism of ∇^η on $\mathbb{S}$. Analogously, for $\Delta^{(0,1)}$, we obtain

$$2\Delta^{(0,1)} = \Delta^\eta_H + i2m\nabla^\eta_\xi - i\sum_{j=1}^m R(e_j, f_j).$$

Thus, we have

$$(1 - \tfrac{i}{2m} d\eta\cdot)\Delta^{(1,0)} + (1 + \tfrac{i}{2m} d\eta\cdot)\Delta^{(0,1)} = \Delta^\eta_H - d\eta \cdot \nabla^\eta_\xi + \frac{1}{2m} d\eta \sum_{j=1}^m R(e_j, f_j).$$

Comparing this with the Schrödinger-Lichnerowicz formula (3.14), we obtain

$$(D_H^\eta)^2 = (1 - \tfrac{i}{2m} d\eta\cdot)\Delta^{(1,0)} + (1 + \tfrac{i}{2m} d\eta\cdot)\Delta^{(0,1)} + \frac{1}{4}\operatorname{scal} - \frac{1}{2m} d\eta \sum_{j=1}^{2m} R(e_j, f_j).$$

It remains to calculate the curvature term. From Lemma 2.2.2, we have

$$R(e_j, f_j)\varphi = \sum_{\alpha,\beta=1}^{2m} R^\eta(e_j, f_j, b_\alpha, b_\beta) b_\alpha \cdot b_\beta \cdot \varphi,$$

where $(b_\alpha)_{j=1}^{2m}$ is a renaming of the basis (e_j, f_j) that "forgets" that it is adapted for notational ease. Thus, using the Bianchi identity for R^η, we obtain

$$\begin{aligned} &R(e_j, f_j)\varphi \\ &= \sum_{\alpha,\beta=1}^{2m} \big(- R^\eta(f_j, b_\alpha, e_j, b_\beta) - R^\eta(b_\alpha, e_j, f_j, b_\beta) + d\eta(e_j, f_j) g(\tau b_\alpha, b_\beta) \\ &\qquad + d\eta(f_j, b_\alpha) g(\tau e_j, b_\beta) + d\eta(b_\alpha, e_j) g(\tau f_j, b_\beta)\big) b_\alpha \cdot b_\beta \cdot \varphi. \end{aligned}$$

For the terms involving the curvature tensor, we use that $R^\eta(\cdot,\cdot,J\cdot,J\cdot) = R^\eta$ (this is an immediate consequence of the facts that ∇^η is adapted and that $g(J\cdot,J\cdot) = g$ on H) and obtain

$$-R^\eta(f_j, b_\alpha, e_j, b_\beta) - R^\eta(b_\alpha, e_j, f_j, b_\beta) = R^\eta(b_\alpha, f_j, f_j, Jb_\beta) + R^\eta(b_\alpha, e_j, e_j, Jb_\beta)$$

and thus

$$\sum_{j=1}^{m} \sum_{\alpha,\beta=1}^{2m} \big(- R^\eta(f_j, b_\alpha, e_j, b_\beta) - R^\eta(b_\alpha, e_j, f_j, b_\beta)\big) b_\alpha b_\beta \varphi = \sum_{\alpha,\beta=1}^{2m} \operatorname{Ric}^\eta(b_\alpha, Jb_\beta) b_\alpha b_\beta \varphi.$$

We will now show that this is the Clifford action of Ric^J. To see this, we consider the following tensor:

$$\operatorname{Ric}^\eta_- = \tfrac{1}{2}(\operatorname{Ric}^\eta(\cdot,\cdot) - \operatorname{Ric}^\eta(J\cdot, J\cdot)).$$

In particular, $\operatorname{Ric}^\eta_-(J\cdot,J\cdot) = -\operatorname{Ric}^\eta_-$ and $\operatorname{Ric}^\eta = \operatorname{Ric}^\eta_+ + \operatorname{Ric}^\eta_-$. Thus, for $X, Y \in H$,

$$\operatorname{Ric}^\eta_-(X, JY) = -\operatorname{Ric}^\eta_-(JX, J^2Y) = \operatorname{Ric}^\eta_-(JX, Y) = \operatorname{Ric}^\eta_-(Y, JX),$$

i.e. $\operatorname{Ric}^J_- = \operatorname{Ric}^\eta_-(\cdot, J\cdot)$ can be interpreted as a symmetric endomorphism of H. Its trace is given by

$$\operatorname{tr}\operatorname{Ric}^J_- = \sum_{j=1}^{m} \operatorname{Ric}^\eta_-(e_j, f_j) - \operatorname{Ric}^\eta_-(f_j, e_j) = 0.$$

Thus, we obtain

$$\begin{aligned}\sum_{\alpha,\beta=1}^{2m} \mathrm{Ric}^{\eta}(b_\alpha, Jb_\beta)b_\alpha b_\beta\varphi &= \sum_{\alpha,\beta=1}^{2m} \mathrm{Ric}^{J}(b_\alpha, b_\beta)b_\alpha b_\beta\varphi + \mathrm{Ric}^{J}_{-}(b_\alpha, b_\beta)b_\alpha b_\beta\varphi \\ &= \mathrm{Ric}^{J}\cdot\varphi + \sum_{\beta=1}^{2m} b_\alpha \,\mathrm{Ric}^{J}_{-}(b_\alpha)\varphi.\end{aligned}$$

The sum involving Ric^{J}_{-} vanishes by (3.9).

For the remaining terms, we have

$$\begin{aligned}&\sum_{j=1}^{m}\sum_{\alpha,\beta=1}^{2m} \big(g(\tau b_\alpha, b_\beta) + d\eta(f_j, b_\alpha)g(\tau e_j, b_\beta) + d\eta(b_\alpha, e_j)g(\tau f_j, b_\beta)\big)b_\alpha\cdot b_\beta\cdot\varphi \\ =&m\sum_{\alpha=1}^{2m} b_\alpha\tau b_\alpha + \sum_{j=1}^{m}\sum_{\alpha=1}^{2m} b_\alpha\big(d\eta(f_j, b_\alpha)\tau e_j + d\eta(b_\alpha, e_j)\tau f_j\big)\cdot\varphi.\end{aligned}$$

The first sum vanishes by (3.9). For the second one, observe that

$$d\eta(f_j, b_\alpha) = -2g(e_j, b_\alpha), \quad d\eta(b_\alpha, e_j) = -2g(f_j, d\eta).$$

This implies that the second sum transforms into a constant multiple of the first and also vanishes. This yields the claimed formula. □

3.3 CR-conformal covariance

Recall that the choice of a contact form that makes a given CR structure (M, H, J) strictly pseudoconvex is not unique but the form can rather be CR-conformally changed, i.e. we can replace one form η by $\tilde{\eta} = e^{2u}\eta$. This change induces a change of the Webster metric that is conformal on H and not conformal on the complement. We now want to investigate how the horizontal Dirac operator associated with the Tanaka-Webster connection changes under such a CR-conformal change of metric.

We begin by considering the change of the Tanaka-Webster connection. We note ∇^η and $\widetilde{\nabla}^\eta = \nabla^{\tilde{\eta}}$ the Tanaka-Webster connections of η and $\tilde{\eta} = e^{2u}\eta$ respectively, g and $\tilde{g}$ the associated Webster metrics and ∇^g and $\widetilde{\nabla}^g = \nabla^{\tilde{g}}$ the Levi-Civita connections of these metrics. We deduce from the Koszul formula that the Levi-Civita connection transforms on H as for a usual conformal change of metric, i.e. we have for any $X, Y, Z \in \Gamma(H)$:

$$g(\widetilde{\nabla}^g_X Y, Z) = g(\nabla^g_X Y, Z) + X(u)g(Y, Z) + Y(u)g(X, Z) - Z(u)g(X, Y).$$

Given an ON basis (b_j) of H for g, the vectors $\widetilde{b_j} = e^{-u}b_j$ form one for $\tilde{g}$. Then, recalling that

$$\tilde{g}(\widetilde{\nabla}^{\eta}_X Y, Z) = \tilde{g}(\widetilde{\nabla}^{g}_X Y, Z) - \tilde{T}(X;Y,Z) + \tfrac{3}{2}\mathfrak{b}\tilde{T}(X,Y,Z),$$

where $\tilde{T}$ is the torsion of $\widetilde{\nabla}^{\eta}$ (considered as a (3,0)-tensor via $\tilde{g}$) and that $T(X;Y,Z) = 0$ for $X, Y, Z \in \Gamma(H)$, we obtain

$$\begin{aligned}
\widetilde{\nabla}^{\eta}_X Y &= \sum_{i=1}^{2m} \tilde{g}(\widetilde{\nabla}^{\eta}_X Y, \tilde{b_j})\tilde{b_j} \\
&= \sum_{i=1}^{2m} e^{2u} g(\widetilde{\nabla}^{g}_X Y, e^{-u}b_j)e^{-u}b_j \\
&= \sum_{i=1}^{2m} g(\widetilde{\nabla}^{g}_X Y, b_j)b_j \\
&= \sum_{i=1}^{2m} g(\nabla^{g}_X Y, b_j)b_j + X(u)g(Y,b_j)b_j + Y(u)g(X,b_j)b_j - b_j(u)g(X,Y)b_j.
\end{aligned}$$

Thus, we have proven the following lemma.

Lemma 3.3.1. *Let (M, H, J, η) be a strictly pseudoconvex CR manifold and $\tilde{\eta} = e^{2u}\eta$. Then, the Tanaka-Webster connections transforms as follows for any $X, Y \in \Gamma(H)$:*

$$\widetilde{\nabla}^{\eta}_X Y = \nabla^{\eta}_X Y + X(u)Y + Y(u)X - g(X,Y)\operatorname{grad}^{\eta}_H u, \tag{3.17}$$

where $\operatorname{grad}^{\eta}_H u$ is the horizontal gradient with respect to g, i.e.

$$\operatorname{grad}^{\eta}_H u = pr^{g}_H(\operatorname{grad}^{g} u) = \sum_{i=1}^{2m} g(\operatorname{grad}^{g} u, b_j)b_j = \sum_{i=1}^{2m} b_j(u)b_j,$$

where pr^{g}_H is the orthogonal (with respect to g) projection onto H and (b_j) is some local orthonormal basis of H.

Now assume that $(M, g = g_\eta)$ is spin with a fixed spin structure $f\colon P_{Spin}(M,g) \to P_{SO}(M,g)$ (we assume that the orientation is given by the volume form $\eta \wedge (d\eta)^m$). We will show that $(M, \tilde{g})$ is then also spin and the associated spinor bundles are isomorphic. We begin by defining an isomorphism between the respective bundles of oriented orthonormal frames:

$$\phi_u\colon P_{SO}(M,g) \longrightarrow P_{SO}(M,\tilde{g}).$$

It is defined as follows: Fix an oriented g-orthonormal basis (b_j) of H. Then $(b_1, ..., b_{2m}, \xi)$ is an oriented orthonormal basis of TM. Setting $\tilde{b}_j = e^{-u}b_j$ and

$\tilde{\xi} = \tilde{\eta}^\natural$ (or, equivalently, defining $\tilde{\xi}$ as the Reeb vector field of $\tilde{\eta}$) then gives a $\tilde{g}$-orthonormal frame. Now, given any g-ON basis (s_j) of TM, any element of this basis can be expressed as

$$s_j = \sum_{k=1}^{2m} a_{j,k} b_k + a_{j,\xi}\xi.$$

We then define $\phi_u(s_1, ..., s_{2m+1}) =: (\tilde{s}_1, ..., \tilde{s}_{2m+1})$ via

$$\tilde{s}_j = \sum_{k=1}^{2m} a_{j,k} \tilde{b}_k + a_{j,\xi}\tilde{\xi}.$$

One may check that ϕ_u is well-defined, i.e. it is independent of the basis (b_j) chosen and maps to $P_{SO}(M, \tilde{g})$. Furthermore, if (s_j) is of type $(v_1, ..., v_{2m}, \xi)$ with (v_j) an ON basis of H, it obviously maps to $(e^{-u}v_1, ..., e^{-u}v_{2m}, \tilde{\xi})$.

One now uses covering theory to prove the following results just as in the usual Riemannian case:

Lemma 3.3.2. *There exists a two-fold covering $\tilde{f} \colon P_{Spin}(M, \tilde{g}) \to P_{SO}(M, \tilde{g})$ and a map $\Psi_u \colon P_{Spin}(M, g) \to P_{Spin}(M, \tilde{g})$ such that*

1. *$(P_{Spin}(M, \tilde{g}), \tilde{f})$ is a spin structure on $(M, \tilde{g})$.*
2. *The diagram*

$$\begin{array}{ccc} P_{Spin}(M,g) & \xrightarrow{\Psi_u} & P_{Spin}(M,\tilde{g}) \\ f\downarrow & & \downarrow\tilde{f} \\ P_{SO}(M,g) & \xrightarrow{\phi_u} & P_{SO}(M,\tilde{g}) \end{array}$$

commutes.

This induces isomorphisms on the associated vector bundles, in particular the spinor and the tangent bundles:

$$\begin{aligned} F_u : \mathbb{S} = P_{Spin}(M, g) \times_{Spin} \Delta_{2m+1} &\longrightarrow \tilde{\mathbb{S}} = P_{Spin}(M, \tilde{g}) \times_{Spin} \Delta_{2m+1} \\ \varphi = [q, v] &\longmapsto [\Psi_u(q), v] =: \tilde{\varphi}. \\ G_u : TM = P_{Spin}(M, g) \times_{Spin} \mathbb{R}^{2m+1} &\longrightarrow TM = P_{Spin}(M, \tilde{g}) \times_{Spin} \mathbb{R}^{2m+1} \\ X = [q, x] &\longmapsto [\Psi_u(q), x] =: \tilde{X}. \end{aligned}$$

Then, by the definition of Clifford multiplication, we obtain

$$\tilde{X} \cdot \tilde{\varphi} = [\Psi_u(q), x] \cdot [\Psi_u(q), v] = [\Psi_u(q), x \cdot v] = \widetilde{X \cdot \varphi} \tag{3.18}$$

for any $X \in TM$. Moreover, any element U of H may always be represented in $TM = P_{Spin}(M,g) \times_{Spin} \mathbb{R}^{2m+1}$ in the following form:

$U = [\hat{s}, (x_1, ..., x_{2m}, 0)]$, where $\hat{s}$ is a lift of $s = (e_1, ..., e_{2m}, \xi) \in P_{SO}(M)$.

Then we know that

$$\begin{aligned} \tilde{U} &= [\Psi_u(\hat{s}), (x_1, ..., x_{2m}, 0)] \\ &= [\tilde{f}(\Psi_u(\hat{s})), (x_1, ..., x_{2m}, 0)] \\ &= [\phi_u(f(\hat{s})), (x_1, ..., x_{2m}, 0)] \\ &= [\tilde{s}, (x_1, ..., x_{2m}, 0)] \\ &= e^{-u} U. \end{aligned} \tag{3.19}$$

The Tanaka-Webster connection induces a connection on the spinor bundle. We now want to calculate the change of this connection under a CR-conformal change of η. We first recall the local formula for spinor connection. Let (b_j) be a local ON basis of H. Hence, $s = (b_1, \dots, b_{2m}, \xi)$ is a local section of $P_{SO}(M,g)$ and we can consider its lift $\hat{s}$ to $P_{Spin}(M,g)$. Then, we can locally represent $\varphi \in \Gamma(\mathbb{S})$ as $\varphi|_U = [\hat{s}, v]$, $v \in C^\infty(U, \Delta_{2m+1})$. The spinor connection is then locally given by

$$\nabla^\eta_X \varphi|_U = [\hat{s}, X(v)] + \tfrac{1}{2} \sum_{\substack{j,k=1 \\ j<k}}^{2m} g(\nabla^\eta_X b_j, b_k) b_j \cdot b_k \cdot \varphi \tag{3.20}$$

because $\nabla(\Gamma(H)) \subset \Gamma(T^*M \otimes H)$.

We now change our pseudo-Hermitian structure conformally. Then, the local formula for the spinor connection combined with the formula for the change of ∇^η yields a formula for the change of the spinor connection and the horizontal Dirac operator.

Theorem 3.3.3. *Let (M^{2m+1}, H, J, η) be a spin strictly pseudoconvex CR manifold and $\tilde{\eta} = e^{2u}\eta$ a CR-conformal change of the contact form. Then, on the spinor bundle we have the following induced changes.*

1. *The spinor connection induced by ∇^η changes as follows:*

$$\tilde{\nabla}^\eta_X \tilde{\varphi} = \widetilde{\nabla^\eta_X \varphi} - \tfrac{1}{2}\left(\tilde{X} \cdot \widetilde{\operatorname{grad}^\eta_H u} \cdot \tilde{\varphi} + g_\eta(X, \operatorname{grad}^\eta_H u)\tilde{\varphi}\right).$$

2. *The horizontal Dirac operator associated with ∇^η is CR-conformally covariant, i.e. it transforms as follows:*

$$\widetilde{D^\eta_H \varphi} = e^{\frac{2m+1}{2}u} D^{\tilde{\eta}}_H \left(e^{-\frac{2m-1}{2}u} \tilde{\varphi}\right).$$

Proof. Concerning the spinor connection, the formula is immediately deduced from (3.17) and (3.20).

We now use this formula to obtain CR-conformal covariance just as one obtains the conformal covariance of the usual Dirac operator (We follow the arguments as they are presented in [Gin09, Proposition 1.3.10] here). Let (b_j) be an orthonormal basis of H for g. For larger expressions, we will write $[X]^\sim$ instead of $\tilde{X}$. Then,

$$\begin{aligned}
D_H^{\tilde{\eta}}\tilde{\varphi} &= \sum_{i=1}^{2m} \tilde{b}_j \cdot \widetilde{\nabla}^{\eta}_{\tilde{b}_j}\tilde{\varphi} \\
&= e^{-u}\sum_{j=1}^{2m} \tilde{b}_j \cdot \widetilde{\nabla}^{\eta}_{b_j}\tilde{\varphi} \\
&= e^{-u}\sum_{j=1}^{2m} \left[b_j \cdot \nabla^{\eta}_{b_j}\varphi\right]^\sim - \frac{e^{-u}}{2}\sum_{j=1}^{2m}\left(\tilde{b}_j \cdot \tilde{b}_j \cdot \widetilde{\operatorname{grad}^{\eta}_H} u \cdot \tilde{\varphi} + g_\eta(b_j, \operatorname{grad}^{\eta}_H u)\tilde{b}_j \cdot \tilde{\varphi}\right) \\
&= e^{-u}\left(\widetilde{D^{\eta}_H\varphi} + m[\operatorname{grad}^{\eta}_H u]^\sim \cdot \tilde{\varphi} - \frac{1}{2}\sum_{j=1}^{2m} g_\eta(b_j, \operatorname{grad}^{\eta}_H u)\tilde{b}_j \cdot \tilde{\varphi}\right).
\end{aligned}$$

Concerning the last sum, we have

$$\sum_{j=1}^{2m} g_\eta(b_j, \operatorname{grad}^{\eta}_H u)\tilde{b}_j = e^{-u}\sum_{j=1}^{2m} g_\eta(b_j, \operatorname{grad}^{\eta}_H u)b_j = e^{-u}\operatorname{grad}^{\eta}_H(u) = [\operatorname{grad}^{\eta}_H u]^\sim.$$

Thus, we obtain

$$D_H^{\tilde{\eta}}\tilde{\varphi} = e^{-u}\left(\widetilde{D^{\eta}_H\varphi} + (m - \tfrac{1}{2})\left[\operatorname{grad}^{\eta}_H u \cdot \varphi\right]^\sim\right)$$

Then, using this, the formula for the product of a function and a spinor under D^{η}_H (cf Lemma 3.1.2) and

$$\operatorname{grad}^{\eta}_H(e^{-\frac{2m-1}{2}u}) = \tfrac{1-2m}{2}e^{-\frac{2m-1}{2}u} \cdot \operatorname{grad}^{\eta}_H u,$$

we deduce that

$$\begin{aligned}
D_H^{\tilde{\eta}}\left(e^{-\frac{2m-1}{2}u}\tilde{\varphi}\right) &= e^{-u}\left(\left[D^{\eta}_H\left(e^{-\frac{2m-1}{2}u}\varphi\right)\right]^\sim + \frac{2m-1}{2}e^{-\frac{2m-1}{2}u}[\operatorname{grad}^{\eta}_H u \cdot \varphi]^\sim\right) \\
&= e^{-u}\left(e^{-\frac{2m-1}{2}u}\widetilde{D^{\eta}_H\varphi} + \tfrac{1-2m}{2} \cdot e^{-\frac{2m-1}{2}u}[\operatorname{grad}^{\eta}_H u \cdot \varphi]^\sim\right. \\
&\qquad \left. + \frac{2m-1}{2}e^{-\frac{2m-1}{2}u}[\operatorname{grad}^{\eta}_H u \cdot \varphi]^\sim\right) \\
&= e^{-\frac{2m+1}{2}u}\widetilde{D^{\eta}_H\varphi}
\end{aligned}$$

□

3.4 Example: S^1-bundles

Contact and CR structures on (the total spaces of) S^1-bundles over almost-Hermitian manifolds provide a large class of examples, these were introduced at the end of Section 1.1. They are particularly useful for studying horizontal Dirac operators as we can compare them with the (normal) Dirac operator on the base manifold. We study general S^1-bundles first, before moving on to spheres in particular.

We begin by comparing the spin structures of base manifold and total space, following the approach of Moroianu [Mor96, chapter 1]. In what follows, we consider a Kähler manifold $(\bar{M}^{2m}, \bar{g}, \bar{J})$ and an S^1-bundle $\pi\colon M^{2m+1} \to \bar{M}$ with metric contact and CR structure (g, J, η) as described at the end of section 1.1 and in Proposition 1.2.5.

Lemma 3.4.1 ([Mor96, chapter 1])**.** *If $(\bar{M}, \bar{g})$ has a spin structure, so does (M, g) and it is given by*

$$P_{Spin}(M) = \left(\pi^* P_{Spin}(\bar{M})\right) \times_{Spin_{2m}} Spin_{2m+1}.$$

Proof. The bundle of oriented orthonormal frames $P_{SO}(M)$ admits a SO_{2m}-reduction given at any point $p \in M$ by

$$(P_H(M))_p = \{\, (b_1, ..., b_{2m}, \xi) \mid (b_j) \text{ ON basis of } H_p\} \subset (P_{SO}(M))_p.$$

The bundle $P_H(M)$ is isomorphic to the pull-back of the bundle of orthonormal frames on $\bar{M}$: $P_H(M) \simeq \pi^* P_{SO}(\bar{M})$. Thus, if $\bar{M}$ is spin, we can induce a spin structure on $P_H(M)$ as follows: We set $P_{Spin,H}(M) = \pi^* P_{Spin}(\bar{M})$, which is a $Spin_{2m}$-principal bundle, and change the type of this principal bundle by setting

$$P_{Spin}(M) = P_{Spin,H}(M) \times_{Spin_{2m}} Spin_{2m+1},$$

which is a two-fold covering of $P_{SO}(M) = P_H(M) \times_{SO_{2m}} SO_{2m+1}$ and indeed a spin-structure on M. $\square$

Using the above structure, where $P_{Spin}(M)$ is an extension of $P_{Spin,H}(M)$, we deduce for the spinor bundle

$$\begin{aligned} \mathbb{S} &= P_{Spin}(M) \times_{Spin_{2m+1}} \Delta_{2m+1} \\ &= P_{Spin,H}(M) \times_{Spin_{2m}} \Delta_{2m} \\ &= \pi^* \bar{\mathbb{S}} \end{aligned}$$

where $\bar{\mathbb{S}}$ denotes the spinor bundle of $\bar{M}$. Concerning spinor fields, we may in particular choose sections constant along the fibres:

Definition. A *projectable spinor* is a spinor field given as $\varphi^* = \varphi \circ \pi$ for some spinor field $\varphi \in \Gamma(\bar{\mathbb{S}})$. We denote the space of projectable spinors by $\Gamma_p(\mathbb{S})$.

The space of all spinor fields on M is then given by

$$\Gamma(\mathbb{S}) = \Gamma_p(\mathbb{S}) \otimes_{C^\infty(\bar{M})} C^\infty(M).$$

Using the definition of Clifford multiplication, we deduce that

$$X^* \cdot \psi^* = (X \cdot \psi)^*.$$

Next, we compare the Tanaka-Webster connection ∇^η on M with the Levi-Civita connection $\nabla^{\bar{g}}$ on $\bar{M}$, both as a derivative on TM and as a spinor derivative. Considering a strictly pseudoconvex CR manifold (M, H, J, η), its CR subbundle H coincides with the horizontal tangent space of the submersion and is thus locally spanned by the horizontal lifts of vector fields on $\bar{M}$. We use this fact to describe the Tanaka-Webster connection on the horizontal distribution.

Proposition 3.4.2. *The Tanaka-Webster connection on a Sasaki manifold given as an S^1-bundle over a Kähler manifold is given by*

$$\begin{aligned} \nabla^\eta_{X^*} Y^* &= (\nabla^{\bar{g}}_X Y)^*, \\ \nabla^\eta_\xi X^* &= 0, \\ \nabla^\eta \xi &= 0, \end{aligned}$$

where X^*, Y^* are horizontal lifts of $X, Y \in \mathfrak{X}(\bar{M})$.

Proof. Recall that

$$g(\nabla^\eta_X Y, Z) = g(\nabla^g_X Y, Z) - T^\eta(X; Y, Z) + \tfrac{3}{2}\mathfrak{b}T^\eta(X, Y, Z), \tag{3.21}$$

where T^η is the torsion of the Tanaka-Webster connection. As T^η is zero when all three arguments are in H and $\Gamma(H)$ is stable under ∇^η, we deduce that

$$g(\nabla^\eta_{X^*} Y^*, Z^*) = g(\nabla^g_{X^*} Y^*, Z^*) \quad \text{and} \quad g(\nabla^\eta_{X^*} Y^*, \xi) = 0, \tag{3.22}$$

where $Z \in \mathfrak{X}(\bar{M})$. Using the relationship between the Levi-Civita-connection on M and $\bar{M}$ (cf. Lemma 1.1.8), we have

$$\nabla^g_{X^*} Y^* = (\nabla^{\bar{g}}_X Y)^* + F^1_{X^*} Y^* = (\nabla^{\bar{g}}_X Y)^* + \tfrac{1}{2} v[X^*, Y^*],$$

where F^1 is the first fundamental tensor of the submersion. Thus, using (3.22), we have

$$g(\nabla^\eta_{X^*} Y^*, Z^*) = g((\nabla^{\bar{g}}_X Y)^* + \tfrac{1}{2} v[X^*, Y^*], Z^*) = g((\nabla^{\bar{g}}_X Y)^*, Z^*)$$

and because $(\nabla^{\bar{g}}_X Y)^*$ is horizontal, we obtain

$$\nabla^\eta_{X^*} Y^* = (\nabla^{\bar{g}}_X Y)^*.$$

Next, using (3.21) again and recalling $3\mathfrak{b}T^\eta = \eta \wedge d\eta$, we deduce

$$g(\nabla^\eta_\xi X^*, Y^*) = g(\nabla^g_\xi X^*, Y^*) - \tfrac{1}{2} d\eta(X^*, Y^*).$$

From the Koszul formula, we obtain

$$2g(\nabla^g_\xi X^*, Y^*) = \xi g(X^*, Y^*) + g([\xi, X^*], Y^*) - g([\xi, X^*], Z^*) - g([X^*, Y^*], \xi).$$

Because the commutator of ξ and a horizontal lift is vertical by Lemma 1.1.8 and $g(X^*, Y^*)$ is invariant in ξ-direction, this reduces to

$$2g(\nabla^g_\xi X^*, Y^*) = -g([X^*, Y^*], \xi) = d\eta(X^*, Y^*).$$

This yields the claim. □

Recall that the spinor derivative $\nabla^\eta \colon \Gamma(\mathbb{S}) \to \Gamma(T^*M \otimes \mathbb{S})$ is given by the local formula

$$\nabla^\eta_X \varphi|_U = [\hat{s}, X(v)] + \frac{1}{2} \sum_{\substack{j,k=1 \\ j<k}}^{2m+1} g(\nabla^\eta_X s_j, s_k) s_j \cdot s_k \cdot \varphi,$$

where s is a section of $P_{SO}(M)|_U$ and $\hat{s}$ its lift to $P_{Spin}(M)|_U$. In particular, we may chose $s = (b_1^*, ..., b_{2m}^*, \xi)$, where the b_j^* are the horizontal lifts of an ON basis (b_j) of $T\bar{M}$. Then, the above formula becomes

$$\nabla^\eta_X \varphi|_U = [\hat{s}, X(v)] + \frac{1}{2} \sum_{\substack{j,k=1 \\ j<k}}^{2m} g(\nabla^\eta_X b_j^*, b_k^*) b_j^* \cdot b_k^* \cdot \varphi \tag{3.23}$$

Now, we have just seen that ∇^η (as a connection on TM) is closely related to $\nabla^{\bar{g}}$. This relationship may be extended to the space of projectable spinors.

Proposition 3.4.3. *Let $\varphi = \psi^*$ be a projectable spinor on a Sasaki manifold (M, g, J, η) which is an S^1-bundle over a Kähler manifold $(\bar{M}, \bar{g}, \bar{J})$ such that $i\eta$ is a connection form. Then the covariant derivatives induced by ∇^η on the spinor bundle $\mathbb{S}$ relates to the one induced on $\bar{\mathbb{S}}$ by $\nabla^{\bar{g}}$ as follows:*

$$\nabla^\eta_{X^*} \psi^* = (\nabla^{\bar{g}}_X \psi)^*$$
$$\nabla^\eta_\xi \psi^* = 0.$$

Proof. We will adapt the local formula (3.23) to the situation of a projectable spinor. We may write a projectable spinor as $\psi^*|_U = [\hat{b}^*, v^*]$ with $b^* = (b_1^*, ..., b_{2m}^*, \xi)$, where b_j^* are horizontal lifts of an ON basis of $T\bar{M}$, and $v^* = v \circ \pi$ with $v \in C^\infty(\pi(U), \Delta_{2m})$. Then, we have $[\hat{b}^*, X^*(v^*)] = [\hat{b}^*, (X(v))^*]$ and we obtain for $X \in \mathfrak{X}(\bar{M})$:

$$\begin{aligned}
\nabla^\eta_{X^*}\psi^*|_U &= [\hat{b}^*, X(v)^*] + \tfrac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^{2m} g(\nabla^\eta_X b_j^*, b_k^*) b_j^* \cdot b_k^* \cdot \psi^* \\
&= [\hat{b}, X(v)]^* + \tfrac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^{2m} \bar{g}(\nabla^{\bar{g}}_X b_j, b_k)(b_j \cdot b_k \cdot \psi)^* \\
&= (\nabla^{\bar{g}}_X \psi)^*|_U.
\end{aligned}$$

Turning to the derivative in the direction of ξ, we note that $\xi(v^*) = 0$ and obtain

$$\begin{aligned}
\nabla^\eta_\xi \psi^*|_U &= \tfrac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^{2m} g(\nabla^\eta_\xi b_j^*, b_k^*) b_j^* \cdot b_k^* \cdot \psi^* \\
&= 0
\end{aligned}$$

and have thus proven everything. □

We can now use the comparison of the spinor derivatives to obtain a comparison of the horizontal Dirac operator associated with ∇^η on M and the (normal) Dirac operator on $\bar{M}$.

Theorem 3.4.4. *Let (M, g, J, η) be a Sasaki manifold given as an S^1 bundle over a spin Kähler manifold $(\bar{M}, \bar{g}, \bar{J})$ such that $i\eta$ is a connection form. Then the Tanaka-Webster operator of M and the Dirac operator of $(\bar{M}, \bar{g})$ (associated with the Levi-Civita connection) are related as follows for a projectable spinor ψ^*:*

$$D_H^\eta \psi^* = (D^{\bar{g}}\psi)^*.$$

In particular, D_H^η stabilises the space of projectable spinors.

Proof. This is an immediate consequence of Proposition 3.4.3, the fact that $X^* \cdot \varphi^* = (X \cdot \varphi)^*$ and the local formulae for D_H^η and $D^{\bar{g}}$. □

This relationship gives us information on the spectrum of the Tanaka-Webster operator.

Proposition 3.4.5. *Let (M, g, J, η) be a $2m+1$-dimensional Sasaki manifold given as an S^1 bundle over a closed spin Kähler manifold $(\bar{M}, \bar{g}, \bar{J})$. Then, the point spectrum of D_H^η is always non-empty. In fact, this operator always admits countably infinitely many distinct eigenvalues.*

When restricted to the projectable spinors, the Tanaka-Webster operator has discrete spectrum tending to infinity and the squares of its eigenvalues are bounded from below by

$$\lambda^2 \geq \begin{cases} \frac{2m+1}{8m} \inf_M \mathrm{scal}^\eta & \text{if } m \equiv 1 \pmod 2 \\ \frac{m}{4(m-1)} \inf_M \mathrm{scal}^\eta & \text{if } m \equiv 0 \pmod 2 \end{cases},$$

where scal^η *is the scalar curvature of* ∇^η.

Proof. As it is well known, the operator $D^{\bar{g}}$ is elliptic and essentially self-adjoint and thus has discrete spectrum. By Theorem 3.4.4, any eigenvalue of $D^{\bar{g}}$ is an eigenvalue of D_H^η. This yields the first statement.

When restricted to projectable spinors, D_H^η completely "coincides" with $D^{\bar{g}}$ and thus the spectra are equal. We may then apply Kirchberg's inequality for the eigenvalues of $D^{\bar{g}}$ on Kähler manifolds, cf. [Kir86]. Hence, we only need to compare scal^η and $\mathrm{scal}^{\bar{g}}$: Using the results from section 1.5, we know that

$$\mathrm{scal}^\eta = \sum_{j,k=1}^{2m} R^\eta(b_k^*, b_j^*, b_j^*, b_k^*),$$

where R^η is the curvature tensor of ∇^η. Thus, using the results of proposition 3.4.2 and the fact that $[b_j^*, b_k^*] = [b_j, b_k]^* + f\xi$, we obtain

$$\begin{aligned}
\mathrm{scal}^\eta &= \sum_{j,k=1}^{2m} R^\eta(b_j^*, b_k^*, b_k^*, b_j^*) \\
&= \sum_{j,k=1}^{2m} g(\nabla^\eta_{b_j^*} \nabla^\eta_{b_k^*} b_k^*, b_j^*) - g(\nabla^\eta_{b_k^*} \nabla^\eta_{b_j^*} b_k^*, b_j^*) - g(\nabla^\eta_{[b_j^*, b_k^*]} b_k^*, b_j^*) \\
&= \sum_{j,k=1}^{2m} g(\nabla^\eta_{b_j^*} (\nabla^{\bar{g}}_{b_k} b_k)^*, b_j^*) - g(\nabla^\eta_{b_k^*} (\nabla^{\bar{g}}_{b_j} b_k)^*, b_j^*) - g(\nabla^\eta_{[b_j, b_k]^*} b_k^*, b_j^*) \\
&= \sum_{j,k=1}^{2m} \bar{g}(\nabla^{\bar{g}}_{b_j} \nabla^{\bar{g}}_{b_k} b_k, b_j) - \bar{g}(\nabla^{\bar{g}}_{b_k} \nabla^{\bar{g}}_{b_j} b_k, b_j) - \bar{g}(\nabla^{\bar{g}}_{[b_j, b_k]} b_k, b_j) \\
&= \mathrm{scal}^{\bar{g}}.
\end{aligned}$$

This yields the claim. □

Remark. We will see in section 4.5 (at least in dimension ≥ 5) that it is in fact a general property of D_H^η that it has an infinite number of discrete eigenvalues. In fact, $(D_H^\eta)^2$ has pure discrete point spectrum and all eigenspaces except possibly the kernel are finite-dimensional.

3.4.1 Application: the spectrum of D_H^η on spheres

We consider a sphere of dimension $4k+3$, $k \in \mathbb{N}_0$, which may be seen as an S^1-bundle $\pi : S^{4k+3} \to \mathbb{C}P^{2k+1}$ (the Hopf fibration), where we equip the base space with the Fubini-Study metric of constant holomorphic sectional curvature 4. We set $\xi(x) = \tilde{J}x$, where $\tilde{J}$ is the standard-complex structure on $\mathbb{R}^{4k+4} \simeq \mathbb{C}^{2k+2}$ and note $\langle\cdot,\cdot\rangle$ the standard scalar product. We thus obtain the standard Sasaki structure on S^{4k+3} as $(M, \langle\cdot,\cdot\rangle, \xi)$. We note that the corresponding strictly pseudoconvex CR structure is given by $H_p = T_pM \cap i \cdot (T_pM)$, $J = \tilde{J}|_H$ and $\eta = \langle \xi, \cdot \rangle$. We observe that $d\pi(\xi) = 0$ and thus the CR structure just described is also induced by the submersion, see also [FIP04, sections 1.2 and 4.3].

It is well known that $\mathbb{C}P^n$ is spin if and only if n is odd and the spectrum of the Riemannian Dirac operator on $\mathbb{C}P^{2k+1}$ has been computed by Seifarth and Semmelmann [SS92] and we will now apply their results.

Corollary 3.4.6. *The point spectrum of the Tanaka-Webster operator of the sphere S^{4k+3} with the standard Sasaki structure, when restricted to the projectable spinors, consists of the values $\pm\sqrt{\lambda_{a,b}}$ and $\pm\sqrt{\mu_{a,b}}$, where:*

$$\begin{aligned}
\lambda_{a,b} &= (a+k)(a+2k+1-b)\\
&\quad b \in \{1, ..., 2k+1\} \textit{ and } a \geq \max\{1, b-k\}\\
\mu_{a,b} &= (a+k+1)(a+2k+1-b)\\
&\quad b \in \{0, ..., 2k\} \textit{ and } a \geq \max\{0, b-k\}
\end{aligned}$$

Proof. The sphere S^{4k+3} is an S^1-bundle over the odd-dimensional complex projective space $\mathbb{C}P^{2k+1}$. We can then apply the results of [SS92] for this space and obtain the same eigenvalues of D_H^η by Theorem 3.4.4. □

The full spectrum on S^3

In the general case, we are unable to extend our computations of the spectrum to the full spinor bundle. Recall that the full spinor bundle is given as $\Gamma_p(\mathbb{S}) \otimes_{C^\infty(\bar{M})} C^\infty(M)$. In order to treat the resulting products of functions and projectable spinors, we could use the product formulae from Lemmas 3.1.2 and (3.1.5) but do not know how to treat the resulting terms. In the case of S^3, we have the following additional information: The spectrum of Δ_H is known in this case (see [Chi06]) and the base space $\mathbb{C}P^1 \simeq S^2$ is again a sphere, which comes with a set of Killing spinors trivialising the spinor bundle. The latter fact has already been used by Bär [Bär96] to compute the spectrum of the standard Riemannian Dirac operator on spheres. The Killing spinors and Δ_H-eigenfunctions allow us to obtain further eigenvalues of $(D^\eta_H)^2$.

By identifying $\mathbb{C}P^1$ with the Fubini-Study metric with the sphere equipped with the metric $\frac{1}{4}g_{round}$, we obtain that the spinor bundle of the complex projective space is trivialised by Killing spinors with Killing number one (or by those with Killing number -1). Thus, the spinor bundle of S^3 admits a trivialisation by horizontal ε–Killing spinors with respect to ∇^η, i.e. spinors that satisfy

$$\nabla^\eta_X \varphi = \varepsilon X \cdot \varphi$$

for any $X \in H$, where $\varepsilon = \pm 1$.

To handle the $d\eta \cdot \varphi$ part in (3.4), we need to understand how the Killing spinors behave under the action of $d\eta$. By Proposition 2.3.1, the spinor bundle decomposes into eigenspaces of $d\eta$. In the three-dimensional case, we have $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_-$, where $d\eta \cdot \varphi = \pm 2i\varphi$, where $\varphi \in \mathbb{S}_\pm$. Thus any spinor decomposes as $\varphi = \varphi_+ + \varphi_-$ with $\varphi_\pm \in \Gamma(\mathbb{S}_\pm)$. Locally, for an adapted ON basis $(b_1, b_2 = Jb_1)$ of H, we can write $d\eta = 2b_1 \wedge b_2$ and thus $d\eta \cdot \varphi = 2b_1 \cdot b_2 \cdot \varphi$ and one sees that

$$d\eta \cdot b_1 \cdot \varphi = 2b_1 \cdot b_2 \cdot b_1 \cdot \varphi = -2b_1 \cdot b_1 \cdot b_2 \varphi = -b_1 \cdot d\eta \cdot \varphi$$

and similarly for b_2 and thus, Clifford multiplication with $d\eta$ anticommutes with Clifford multiplication with elements of H (alternatively, one can use the projections p_{10} and p_{01} and use that p_{10} vanishes on $\mathbb{S}_+$ and p_{01} on $\mathbb{S}_-$). Using this and the fact that $d\eta$ is parallel under the Tanaka-Webster connection, we deduce for any horizontal ε-Killing spinor φ that

$$\underbrace{\nabla^\eta_X \varphi_+}_{\in \mathbb{S}_+} + \underbrace{\nabla^\eta_X \varphi_-}_{\in \mathbb{S}_-} = \nabla^\eta_X \varphi = \varepsilon X \cdot \varphi = \varepsilon \underbrace{X \cdot \varphi_+}_{\in \mathbb{S}_-} + \varepsilon \underbrace{X \cdot \varphi_-}_{\in \mathbb{S}_+}$$

for any $X \in \Gamma(H)$ and thus

$$\nabla^\eta_X \varphi_\pm = \varepsilon X \cdot \varphi_\mp \quad \text{for any } X \in \Gamma(H)$$

and therefore

$$D^\eta_H \varphi_\pm = -2\varepsilon \varphi_\mp. \tag{3.24}$$

Combining this with (3.4), we have

$$\begin{aligned}(D^\eta_H)^2(f\varphi_\pm) &= f(D^\eta_H)^2\varphi_\pm - 2\nabla^\eta_{\mathrm{grad}_H f}\varphi_\pm + (\Delta_H f)\varphi_\pm - \xi(f) d\eta \cdot \varphi_\pm \\ &= 4f\varphi_\pm - 2\varepsilon(\mathrm{grad}_H f) \cdot \varphi_\mp + (\Delta_H f)\varphi \mp 2i\xi(f)\varphi_\pm.\end{aligned}$$

In order to deal with the $\operatorname{grad}_H f \cdot \varphi$ part, we change the spinor $f\varphi_\pm$ by adding $\operatorname{grad}_H f \cdot \varphi_\mp$. Under the Dirac operator, this product behaves as follows: By (3.3), we have

$$\begin{aligned} D_H^\eta(\operatorname{grad}_H f \cdot \varphi_\pm) &= -\operatorname{grad}_H f \cdot D_H^\eta \varphi_\pm - 2\nabla^\eta_{\operatorname{grad}_H f}\varphi_\pm + (\Delta_H f - \xi(f)d\eta)\cdot\varphi_\pm \\ &= 2\varepsilon \operatorname{grad}_H f \cdot \varphi_\mp - 2\varepsilon \operatorname{grad}_H f \cdot \varphi_\mp + (\Delta_H f)\varphi_\pm \mp 2i\xi(f)\varphi_\pm \\ &= (\Delta_H f \mp 2i\xi(f))\varphi_\pm \end{aligned} \tag{3.25}$$

We will compute the spectrum and eigenfunctions of Δ_H and then return to this equation.

The spectrum of the Sub-Laplacian

The spectrum of the Sub-Laplacian has been calculated by Chiu in [Chi06]. We give a calculation in real coordinates here. Describing every point on S^3 by the real coordinates (x_0, y_0, x_1, y_1) of the surrounding space, the geometric data has the following form, where we omit the restriction to S^3:

$$\begin{aligned} \eta &= x_0 dy_0 - y_0 dx_0 + x_1 dy_1 - y_1 dx_1, \\ \xi &= x_0 \partial y_0 - y_0 \partial x_0 + x_1 \partial y_1 - y_1 \partial x_1, \\ d\eta &= 2(dx_0 \wedge dy_0 + dx_1 \wedge dy_1), \\ X_1 &= x_1 \partial x_0 - y_1 \partial y_0 - x_0 \partial x_1 + y_0 \partial y_1, \\ X_2 &= y_1 \partial x_0 + x_1 \partial y_0 - y_0 \partial x_1 - x_0 \partial y_1, \end{aligned}$$

where X_1, X_2 form an adapted ON basis of H at each point and together with ξ they furnish a pointwise ON basis of TS^3. On may easily check that the Sub-Laplacian and the Laplacian are given by

$$\begin{aligned} \Delta_H f &= -X_1(X_1(f)) - \operatorname{div}(X_1)X_1(f) - X_2(X_2(f)) - \operatorname{div}(X_2)X_2(f) \\ \Delta^{S^3} f &= -X_1(X_1(f)) - \operatorname{div}(X_1)X_1(f) - X_2(X_2(f)) - \operatorname{div}(X_2)X_2(f) - \xi(\xi(f)), \end{aligned}$$

where we used that $\operatorname{div}^g(\xi) = 0$, cf Corollary 1.1.7. Thus, their difference is given by

$$(\Delta^{S^3} - \Delta_H)f = -\xi(\xi(f)).$$

In euclidean coordinates, this difference becomes

$$\begin{aligned} -\xi\xi(f) = &-\sum_{j=0}^{1}\left(-x_j \tfrac{\partial f}{\partial x_j} - y_j \tfrac{\partial f}{\partial y_j} + x_j^2 \tfrac{\partial^2 f}{\partial y_j^2} + y_j^2 \tfrac{\partial^2 f}{\partial x_j^2} - 2x_j y_j \tfrac{\partial^2 f}{\partial x_j \partial y_j}\right) \\ &- 2\left(x_0 x_1 \tfrac{\partial^2 f}{\partial y_0 \partial y_1} + y_0 y_1 \tfrac{\partial^2 f}{\partial x_0 \partial x_1} - x_0 y_1 \tfrac{\partial^2 f}{\partial y_0 \partial x_1} - y_0 x_1 \tfrac{\partial^2 f}{\partial x_0 \partial y_1}\right). \end{aligned}$$

The full Laplacian in turn compares to the Laplacian on $\mathbb{R}^4$ as follows (cf [BGM71, chapitre II, (G.V.22)])

$$(\Delta^{\mathbb{R}^4} f)|_{S^3} = \Delta^{S^3}(f|_{S^3}) - \left(\frac{\partial^2 f}{\partial r^2}\right)|_{S^3} - 3\left(\frac{\partial f}{\partial r}\right)|_{S^3},$$

where $(r, \theta) \in (0, \infty) \times S^3$ are polar coordinates.

Now, we introduce a family of functions for which we can compute $\Delta_H f$. We use multiindex notation, i.e. for $\alpha = (\alpha_0, \alpha_1)$, we let $|\alpha| = \alpha_0 + \alpha_1$, $x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1}$. Define $\tilde{f}_{\alpha,\beta} \in C^\infty(\mathbb{C}^2)$ to be given by

$$\begin{aligned}\tilde{f}_{\alpha,\beta}(x_0, y_0, x_1, y_1) &= (x_0 + iy_0)^{\alpha_0}(x_0 - iy_0)^{\beta_0}(x_1 + iy_1)^{\alpha_1}(x_1 - iy_1)^{\beta_1} \\ &= r^{(|\alpha|+|\beta|)}(\hat{x}_0 + i\hat{y}_0)^{\alpha_0}(\hat{x}_0 - i\hat{y}_0)^{\beta_0}(\hat{x}_1 + i\hat{y}_1)^{\alpha_1}(\hat{x}_1 - i\hat{y}_1)^{\beta_1},\end{aligned}$$

where the hats mean dividing by the euclidean norm of (x, y), and let $f_{\alpha,\beta}$ be its restriction to the sphere. We denote the associated function spaces by

$$\tilde{P}_4^{p,q} = \operatorname{span}\left\{ \tilde{f}_{\alpha,\beta} \;\middle|\; |\alpha| = p, |\beta| = q \right\}, \quad P_4^{p,q} = \operatorname{span}\{ f_{\alpha,\beta} \mid |\alpha| = p, |\beta| = q \}.$$

Then, we have

$$\begin{aligned}&\Delta^{S^3}(f_{\alpha,\beta}) = (\Delta^{\mathbb{R}^4}\tilde{f}_{\alpha,\beta})|_{S^3} - (|\alpha| + |\beta|)(|\alpha| + |\beta| + 2)f_{\alpha,\beta}, \\ &\xi(\xi(f_{\alpha,\beta})) = -|\alpha|^2 - |\beta|^2 + 2|\alpha| \cdot |\beta|.\end{aligned}$$

Thus, if we let $\tilde{f}$ be a *bigraded spherical harmonic* of type (p, q), i.e. a linear combination of functions $\tilde{f}_{\alpha,\beta}$ with $|\alpha| = p$, $|\beta| = q$ that is harmonic (on $\mathbb{R}^4$) and f its restriction, we obtain

$$\Delta_H f = (4pq + 2p + 2q)f. \tag{3.26}$$

Furthermore, for such a function, we have

$$\xi(f) = i(p - q)f. \tag{3.27}$$

It remains to check that we have thus found all eigenvalues and determine the dimensions of the corresponding eigenspaces.

Lemma 3.4.7. *The spaces*

$$H_4^{p,q} = \left\{ f_{\alpha,\beta} \;\middle|\; |\alpha| = p, |\beta| = q,\, \Delta^{\mathbb{R}^4} f = 0 \right\}$$

for $p, q \in \mathbb{N}$ span $C^\infty(S^3)$ and the dimensions are given by

$$\dim H^{p,q} = p + q + 1.$$

Proof. We begin by noting that all elements of $\tilde{f}_{\alpha,\beta}$ are (complex) linear combinations of homogeneous polynomials of degree $|\alpha|+|\beta|$, i.e. $\tilde{P}^{p,q}$ is a linear subspace of the space of homogeneous polynomials $\tilde{P}^{p+q}$ and in fact

$$\tilde{P}^k = \bigoplus_{p+q=k} \tilde{P}^{p,q}. \tag{3.28}$$

Denoting by $\tilde{H}^k$ the subspace of harmonic functions of $\tilde{P}^k$, we have the following decomposition (cf [BGM71, lemme C.I.2]):

$$\tilde{P}^k = \tilde{H}^k \oplus \|x\|^2 \tilde{H}^{k-2} \oplus \ldots \oplus \|x\|^{2l} \tilde{H}^{k-2l} \quad (l = [\tfrac{k}{2}]),$$

where $\|x\|$ denotes the euclidean norm of $x = (x_0, y_0, x_1, y_1)$. Now, let $f \in \tilde{P}^{p,q} \subset \tilde{P}^{p+q}$, then f admits a decomposition $f = f_0 + f_1 + \ldots + f_l$, where $f_j \in \|x\|^{2j}\tilde{H}^{p+q-2j}$. Noting that $\|x\|^2 = (x_0+iy_0)(x_0-iy_0)+(x_1+iy_1)(x_1-iy_1)$, one deduces that f_j must be in $\|x\|^{2j}\tilde{H}^{p-j,q-j}$ and thus

$$\tilde{P}^{p,q} = \tilde{H}^{p,q} \oplus \|x\|^2 \tilde{H}^{p-1,q-1} \oplus \ldots \oplus \|x\|^{2l} \tilde{H}^{p-l,q-l} \quad (l = \min\{p,q\}).$$

Thus, when restricting to the sphere, we have

$$P^{p,q} = H^{p,q} \oplus H^{p-1,q-1} \oplus \ldots \oplus H^{p-l,q-l} \quad (l = \min p, q), \tag{3.29}$$

i.e. the linear hull of all $H^{p,q}$ is the same as of the $P^{p,q}$ and, using (3.28), the same as the hull of the spaces of homogeneous polynomials, which are known to span the space of smooth functions on the sphere. Furthermore, from (3.29), we deduce

$$\dim H^{p,q} = \dim P^{p,q} - \dim P^{p-1,q-1} = \dim \tilde{P}^{p,q} - \dim \tilde{P}^{p-1,q-1}$$

The spaces $\tilde{P}^{p,q}$ are spanned by the functions $f_{\alpha,\beta}$, i.e their dimensions are given by the possible choices of two multiindices with absolute values p and q. Thus, we obtain

$$\dim H^{p,q} = \binom{p+1}{1}\binom{q+1}{1} - \binom{p}{1}\binom{q}{1} = (p+1)(q+1) - pq = p+q+1.$$

□

Summing up, we have the following result.

Proposition 3.4.8 ([Chi06, p.91]). *The spectrum of the Sub-Laplacian on the three-dimensional sphere with the standard Sasaki structure is given by*

$$\mathrm{spec}(\Delta_H) = \{2k \mid k \in \mathbb{N}\},$$

the corresponding eigenfunctions are smooth and form an L^2 basis of $C^\infty(S^3)$, and the multiplicities are given by

$$m(2k) = \sum_{2pq+p+q=k} p+q+1$$

The spectrum of the Tanaka-Webster operator

Now, let $\bar{\varphi}$ be an ε-Killing spinor over $\mathbb{C}P^1$, $\varphi = (\bar{\varphi})^* = \varphi_+ + \varphi_-$ the associated projectable spinor on S^3, and f an eigenfunction of Δ_H associated with the eigenvalue $\lambda = 4pq + 2p + 2q$ (and thus, $\xi(f) = i(p-q)f$). Setting $\hat{\lambda}_\pm = \lambda \pm 2(p-q)$ and using (3.25), we have

$$D_H^\eta(\mathrm{grad}_H\, f\varphi_\pm) = (\Delta_H f \mp 2i\xi(f))\varphi_\pm = \hat{\lambda}_\pm f\varphi_\pm$$

and thus

$$\begin{aligned}(D_H^\eta)^2(\mathrm{grad}_H\, f\varphi_\pm) &= \hat{\lambda}_\pm \left(D_H^\eta(f\varphi_\pm)\right)\\ &= \hat{\lambda}_\pm \left(fD_H^\eta\varphi_\pm + \mathrm{grad}_H\, f\cdot\varphi_\pm\right)\\ &= \hat{\lambda}_\pm \left(-2\varepsilon f\varphi_\mp + \mathrm{grad}_H\, f\cdot\varphi_\pm\right)\end{aligned}$$

and by (3.4) and (3.24),

$$\begin{aligned}(D_H^\eta)^2 f\varphi_\pm &= 4f\varphi_\pm - 2\varepsilon\,\mathrm{grad}_H\, f\varphi_\mp + (\hat{\lambda}_\pm f)\varphi_\mp\\ &= (4+\hat{\lambda}_\pm)f\varphi_\pm - 2\varepsilon\,\mathrm{grad}_H\, f\varphi_\mp.\end{aligned}$$

Combining the two, we have

$$(D_H^\eta)^2(f\varphi_+ + \mu\,\mathrm{grad}_H\, f\cdot\varphi_-) = (4+\hat{\lambda}_+ - 2\varepsilon\hat{\lambda}_-\mu)f\varphi_+ + (\hat{\lambda}_-\mu - 2\varepsilon)\,\mathrm{grad}_H\, f\cdot\varphi_-, \tag{3.30}$$

$$(D_H^\eta)^2(f\varphi_- + \mu\,\mathrm{grad}_H\, f\cdot\varphi_+) = (4+\hat{\lambda}_- - 2\varepsilon\hat{\lambda}_+\mu)f\varphi_- + (\hat{\lambda}_+\mu - 2\varepsilon)\,\mathrm{grad}_H\, f\cdot\varphi_+. \tag{3.31}$$

We concentrate on the first equation, the calculations for the second one are analogous. The spinor $f\varphi_+ + \mu\,\mathrm{grad}_H\, f\varphi_-$ will give an eigenspinor if either $\mathrm{grad}_H\, f\varphi_\pm$ is a constant multiple of $f\varphi_\mp$ or the right hand side can be written as $\nu(f\varphi_+ + \mu\,\mathrm{grad}_H\, f\varphi_-)$. We will consider the first case later and focus on the second one. If $\mu = 0$ we come back to the first case. Otherwise, this is equivalent to

$$4 + \hat{\lambda}_+ - 2\varepsilon\mu\hat{\lambda}_- = \frac{\hat{\lambda}_-\mu - 2\varepsilon}{\mu}$$

which becomes

$$-2\varepsilon\hat{\lambda}_-\mu^2 + (4+\hat{\lambda}_+ - \hat{\lambda}_-)\mu + 2\varepsilon = 0.$$

This quadratic equation has the solutions

$$\mu_{1,2} = \frac{-(4+\hat{\lambda}_+ - \hat{\lambda}_-) \pm \sqrt{(4+\hat{\lambda}_+ - \hat{\lambda}_-)^2 + 16\hat{\lambda}_-}}{-4\varepsilon\hat{\lambda}_-}.$$

Recalling that $\lambda = 4pq + 2p + 2q$, we see that $\hat{\lambda}_+ = 4pq + 4p$ and $\hat{\lambda}_- = 4pq + 4q$ and thus $\hat{\lambda}_+ - \hat{\lambda}_- = 4(p - q)$. Hence,

$$\mu_{1,2} = \frac{q - p - 1 \pm (1 + p + q)}{-\varepsilon\hat{\lambda}_-}$$

and thus, the associated eigenvalues of $(D_H^\eta)^2$ are

$$\begin{aligned}
\lambda_{p,q}^{+,+} &= 4 + \hat{\lambda}_+ - 2\varepsilon\hat{\lambda}_- \frac{q - p - 1 + (1 + p + q)}{-\varepsilon\hat{\lambda}_-} \\
&= 4 + 4pq + 4p + 2(2q) = 4 + 4pq + 4p + 4q, \\
\lambda_{p,q}^{+,-} &= 4 + \hat{\lambda}_+ - 2\varepsilon\hat{\lambda}_- \frac{q - p - 1 - (1 + p + q)}{-\varepsilon\hat{\lambda}_-} \\
&= 4 + 4pq + 4p + 2(-2 - 2p) = 4pq.
\end{aligned}$$

Analogously, we obtain eigenspinors of the form $f\varphi_- + \mu \operatorname{grad}_H f\varphi_+$ and the associated eigenvalues

$$\begin{aligned}
\lambda_{p,q}^{-,+} &= 4 + \hat{\lambda}_- - 2\varepsilon\hat{\lambda}_+ \frac{p - q - 1 + (1 + p + q)}{-\varepsilon\hat{\lambda}_+} \\
&= 4 + 4pq + 4q + 2(2p) = 4 + 4pq + 4p + 4q, \\
\lambda_{p,q}^{-,-} &= 4 + \hat{\lambda}_- - 2\varepsilon\hat{\lambda}_+ \frac{q - p - 1 - (1 + p + q)}{-\varepsilon\hat{\lambda}_+} \\
&= 4 + 4pq + 4q + 2(-2 - 2q) = 4pq.
\end{aligned}$$

What remains to check is that the associated eigenspaces are not empty, i.e. that the eigenspinors are nonzero. Again, we treat only the case $f\varphi_+ + \mu \operatorname{grad}_H f\varphi_-$ in detail, the other case follows analogously. Before we start the discussion, recall that the other option for obtaining an eigenspinor in (3.30), (3.31) was that $\mu \operatorname{grad}_H f\varphi_\pm$ was a constant multiple of $f\varphi_\mp$. Thus, if we consider whether $f\varphi_+ + \mu \operatorname{grad}_H f\varphi_-$ can become zero with a general μ, we will at the same time treat this case also. If such a spinor were zero, then its its derivatives would be zero as well and in particular,

$$\begin{aligned}
0 &= \nabla_\xi^\eta (f\varphi_+ + \mu \operatorname{grad}_H f\varphi_-) \\
&= \xi(f)\varphi_+ + f\nabla_\xi^\eta\varphi_+ + \mu(\nabla_\xi^\eta \operatorname{grad}_H f) \cdot \varphi_- + \mu \operatorname{grad}_H f \cdot \nabla_\xi^\eta\varphi_-.
\end{aligned}$$

As $\varphi_\pm$ are projectable spinors, their ξ-derivatives vanish. Moreover, we know that $\xi(f) = i(p - q)f$ and thus

$$0 = i(p - q)f\varphi_+ + \mu(\nabla_\xi^\eta \operatorname{grad}_H f) \cdot \varphi_-.$$

Using the original equation $f\varphi_+ + \mu\,\mathrm{grad}_H f \varphi_- = 0$ again, this implies that

$$0 = (-i(p-q)\mu\,\mathrm{grad}_H f + \mu(\nabla^\eta_\xi \mathrm{grad}_H f)) \cdot \varphi_-.$$

We deduce from (3.24) that if φ_- vanishes, so does φ. Thus, the above equation implies that for some constant α,

$$\mathrm{grad}_H f = \alpha \nabla^\eta_\xi \mathrm{grad}_H f.$$

We will now show that this is not possible.

We will calculate the horizontal gradient and its ξ-derivative of functions $f_{\alpha,\beta}$ in Euclidean coordinates. Recall that $X_1 = x_1\partial x_0 - y_1\partial y_0 - x_0\partial x_1 + y_0\partial y_1$ and $X_2 = y_1\partial x_0 + x_1\partial y_0 - y_0\partial x_1 - x_0\partial y_1$ form a local ON basis of H. Thus

$$\mathrm{grad}_H f = X_1(f)X_1 + X_2(f)X_2$$

and

$$\nabla^\eta_\xi(\mathrm{grad}_H f) = \xi(X_1(f))X_1 + X_1(f)\nabla^\eta_\xi X_1 + \xi(X_2(f))X_2 + X_2(f)\nabla^\eta_\xi X_2.$$

Let $f(x,y) = f_{\alpha,\beta}(x_0,y_0,x_1,y_1) = (x_0+iy_0)^{\alpha_0}(x_0-iy_0)^{\beta_0}(x_1+iy_1)^{\alpha_1}(x_1-iy_1)^{\beta_1}$. In what follows, we leave α,β fixed and denote only the changed indices, i.e.

$$f_{\alpha_0-1,\beta_1+1} = f_{\alpha_0-1,\alpha_1,\beta_0,\beta_1+1} = (x_0+iy_0)^{\alpha_0-1}(x_0-iy_0)^{\beta_0}(x_1+iy_1)^{\alpha_1}(x_1-iy_1)^{\beta_1+1}.$$

Then,

$$\begin{aligned}
X_1(f) &= \alpha_0 f_{\alpha_0-1,\beta_1+1} + \beta_0 f_{\beta_0-1,\alpha_1+1} - \alpha_1 f_{\beta_0+1,\alpha_1-1} - \beta_1 f_{\alpha_0+1,\beta_1-1}\\
X_2(f) &= i\alpha_0 f_{\alpha_0-1,\beta_1+1} - i\beta_0 f_{\beta_0-1,\alpha_1+1} - i\alpha_1 f_{\beta_0+1,\alpha_1-1} + i\beta_1 f_{\alpha_0+1,\beta_1-1}.
\end{aligned}$$

As $\xi(f) = i(|\alpha|-|\beta|)f$, we obtain

$$\begin{aligned}
\xi(X_1(f)) &= i(|\alpha|-|\beta|-2)\left(\alpha_0 f_{\alpha_0-1,\beta_1+1} - \alpha_1 f_{\beta_0+1,\alpha_1-1}\right)\\
&\quad + i(|\alpha|-|\beta|+2)\left(\beta_0 f_{\beta_0-1,\alpha_1+1} - \beta_1 f_{\alpha_0+1,\beta_1-1}\right),\\
\xi(X_2(f)) &= (|\alpha|-|\beta|-2)\left(-\alpha_0 f_{\alpha_0-1,\beta_1+1} + \alpha_1 f_{\beta_0+1,\alpha_1-1}\right)\\
&\quad + (|\alpha|-|\beta|+2)\left(\beta_0 f_{\beta_0-1,\alpha_1+1} - \beta_1 f_{\alpha_0+1,\beta_1-1}\right).
\end{aligned}$$

Next, we calculate the ξ-derivatives of X_j. As X_j is a local ON basis, $g(\nabla^\eta_\xi X_j, X_j) = 0$ and it remains to calculate

$$\begin{aligned}
g(\nabla^\eta_\xi X_1, X_2) &= g(\nabla^g_\xi X_1, X_2) - T(\xi; X_1, X_2) + \tfrac{3}{2}\eta\wedge d\eta(\xi, X_1, X_2)\\
&= g(\xi(X_1), X_2) - d\eta(X_1,X_2) + \tfrac{3}{2}d\eta(X_1,X_2)\\
&= g((-y_1,-x_1,-y_0,x_0)^T, X_2) + 1\\
&= x_0^2 + y_0^2 - x_1^2 - y_1^2 + 1,\\
g(\nabla^\eta_\xi X_2, X_1) &= g(\xi(X_2), X_1) - T(\xi; X_2, X_1) + \tfrac{3}{2}\eta\wedge d\eta(\xi, X_2, X_1)\\
&= g((x_1,-y_1,-x_0,y_0)^T, X_1) - 1\\
&= -x_0^2 - y_0^2 + x_1^2 + y_1^2 - 1.
\end{aligned}$$

Using that we're on S^3, we obtain

$$g(\nabla^\eta_\xi X_1, X_2) = 2x_0^2 + 2y_0^2 = 2(x_0 + iy_0)(x_0 - iy_0),$$
$$g(\nabla^\eta_\xi X_2, X_1) = -2x_0^2 - 2y_0^2 = -2(x_0 + iy_0)(x_0 - iy_0).$$

Now, writing $\operatorname{grad}_H f$ and $\nabla^\eta_\xi(\operatorname{grad}_H f)$ in the basis X_1, X_2, we obtain that for some constant c,

$$\xi(X_1(f)) - 2(x_0 + iy_0)(x_0 - iy_0)X_2(f) = cX_1(f),$$
$$\xi(X_2(f)) + 2(x_0 + iy_0)(x_0 - iy_0)X_1(f) = cX_2(f).$$

Let f be of total degree (as a polynomial) $p+q$. Then, from the above calculations, we see that both $X_j(f)$ and $\xi(X_j(f))$ still have the same total degree, whereas $(x_0 + iy_0)(x_0 - iy_0)X_2(f)$ has total degree $p + q + 2$. Thus, we obtain that $(x_0 + iy_0)(x_0 - iy_0)X_j(f)$ must vanish and therefore, so must $X_j(f)$. However, $\Delta_H f = X_1(X_1(f)) + X_2(X_2(f))$ and thus, $p = q = 0$. In this case, f is a constant and $\operatorname{grad}_H f = 0$ and out of the eigenvalues $\lambda^{\pm;\pm}_{0,0}$ we are only left with the eigenvalue 4 which comes from the fact that $\varphi_\pm$ itself is an eigenspinor of eigenvalue 4 by (3.24). Summing up, we have the following result on the spectrum of $(D^\eta_H)^2$.

Theorem 3.4.9. *Let D^η_H be the horizontal Dirac operator associated with the Tanaka-Webster connection on S^3 equipped with its standard Riemannian and CR structure. Then, the following values are eigenvalues of $(D^\eta_H)^2$.*

$$\begin{aligned} \lambda_k &= k^2 && k \in \mathbb{N}_0 \\ \lambda^+_{p,q} &= 4pq && p, q \in \mathbb{N}_0, p + q \neq 0 \\ \lambda^-_{p,q} &= 4(1 + pq + p + q) && p, q \in \mathbb{N}_0. \end{aligned}$$

The eigenvalues λ_k come from projectable spinors and the eigenvalues $\lambda^\pm_{p,q}$ come from spinors of the type $f\varphi_+ + \mu \operatorname{grad}_H f \cdot \varphi_-$, where $\varphi_\pm$ are the parts in the $d\eta$-eigenspaces $\mathbb{S}_\pm$ of the lift to S^3 of a Killing spinor on $\mathbb{C}P^1$ and f is an eigenfunction of Δ_H.

Remark. Unfortunately, we have been unable to determine whether we have thus obtained the complete spectrum of $(D^\eta_H)^2$. However, we do already see some interesting properties of the spectrum: We have infinitely many discrete eigenvalues tending to infinity. On the other hand, we obtain the eigenvalue zero "infinitely often" as $\lambda^+_{p,0}$ and $\lambda^+_{0,q}$ for any $p, q \in \mathbb{N}$. We write infinitely often in quotation marks because we have not proven whether the associated eigenspinors are all linearly independent.

In the next chapter, we will see that this is the general form of the spectrum of the Tanaka-Webster operator, although we will only be able to prove this in dimensions 5 and higher.

3.5 Example: Compact quotients of the Heisenberg group

In this section, we discuss another example where the spectrum of the Tanaka-Webster operator can be explicitly calculated: compact quotients of the Heisenberg group. The spectrum of the (horizontal) Dirac operator on these homogeneous spaces can be computed via representation theory. This has been used to calculate the spectrum of the "normal" Dirac operator on a number of homogeneous spaces. The general theory for Dirac spectra on homogenous spaces does not readily carry over to other connections than the Levi-Civita connection, but in the case of homogeneous manifolds $\Gamma\backslash G$, where $\Gamma < G$ is a discrete subgroup of a nilpotent group, similar calculations as for D^g are possible. The case of compact quotients of the Heisenberg group was treated (for D^g) by C. Bär in his PhD thesis [Bär90] in general dimensions and by B. Ammann and Bär in [AB98] for dimension three. This was picked up and extended to other groups by I. Kath and O. Ungermann for horizontal Dirac operators in [KU13]. In particular, they calculated the spectrum of a horizontal Dirac operator for quotients of the three-dimensional Heisenberg group. In [Has14, section 4.3], S. Hasselmann used the same technique to calculate the spectrum of a horizontal Dirac operator on quotients of groups of type $\mathcal{H}^m \times \mathbb{R}^n$. In our calculations, we make use of the original results of [AB98] and [Bär90].

We recall the structure of the Heisenberg group as we introduced it in Examples 1.1.1 and 1.1.4. The $2m+1$-dimensional Heisenberg group $\mathcal{H}^m$ is topologically $\mathbb{R}^{2m+1}$ with group structure defined by

$$(x, y, z) \cdot (\hat{x}, \hat{y}, \hat{z}) = (x + \hat{x}, y + \hat{y}, z + \hat{z} + \langle x, \hat{y} \rangle),$$

where $x, \hat{x}, y, \hat{y} \in \mathbb{R}^m$ and $z \in \mathbb{R}$. Its contact and CR structure are given as follows: The horizontal distribution is spanned by the left-invariant vector fields

$$X_j = \partial_{x_j} \quad \text{and} \quad Y_j = \partial_{y_j} + x_j \partial_z$$

and is the kernel of the left-invariant one-form $\eta = 2(dz - \sum_{j=1}^m x_j dy_j)$. The Reeb vector field is $\xi = \frac{1}{2}\partial_z$. The almost-complex structure is given by

$$JX_j = -Y_j, \quad JY_j = X_j.$$

Thus, (Y_j, X_j) form an adapted orthonormal (with respect to the Webster metric) basis of H and (Y_j, X_j, ξ) an orthonormal basis of $T\mathcal{H}^m$. The vector fields $X_1, Y_1, Z = 2\xi$ correspond to the left-invariant extensions of the vectors $X, Y, Z \in \mathfrak{g}$ of [AB98] in the case $m = 1$. As Ammann and Bär define their metric by requiring $-dX, -dY, T^{-1}Z$ to be orthonormal for some $d, T \in \mathbb{R}^+$, the Webster metric of the CR structure is their Riemannian metric for $d = 1, T = 2$.

As we have a global orthonormal basis, the bundle of orthonormal frames on $\mathcal{H}^m$ is trivial: $P_{SO}(\mathcal{H}^m) = \mathcal{H}^m \times SO_{2m+1}$. Thus, we obtain a trivial spin structure

$P_{Spin}(\mathcal{H}^m) = \mathcal{H}^m \times Spin_{2m+1}$ and the corresponding spinor bundle is also trivial: $\mathbb{S} = \mathcal{H}^m \times \Delta_{2m+1}$.

As the Heisenberg group is noncompact, we will instead consider compact quotients when calculating the spectrum of the horizontal Dirac operator. Consider the lattice

$$\Gamma_r = \left\{ (r \cdot x, y, z) \in \mathcal{H}^1 \;\middle|\; x, y, z \in \mathbb{Z} \right\},$$

where $r \in \mathbb{N}^m$ such that r_j divides r_{j+1} and $r \cdot x = (r_1 x_1, \ldots, r_m x_m)$. Then, $\Gamma_r < \mathcal{H}^m$ is uniformly discrete and cocompact. Thus, the quotient $M = \Gamma_r \backslash \mathcal{H}^m$ is a compact manifold. As the CR and Riemannian structures on $\mathcal{H}^m$ are left-invariant, they descend to M. As for the spin structures, any homomorphism $\varepsilon\colon \Gamma_r \to \mathbb{Z}_2 = \{\pm 1\}$ induces a spin structure on M via

$$P_{Spin,\varepsilon}(M) = \mathcal{H}^m \times_\varepsilon Spin_{2m+1},$$

where $g \in \Gamma_r$ acts on $\mathcal{H}^m$ by left multiplication and on $Spin_{2m+1}$ by multiplication with $\varepsilon(g)$. It may be shown that any such homomorphism ε is defined by $\delta = (\delta_1, \ldots, \delta_{2m+1}) \in (\mathbb{Z}_2)^{2m+1}$:

$$\varepsilon(rx, y, z) = \delta_1^{x_1} \delta_2^{y_1} \cdots \delta_{2m-1}^{x_m} \delta_{2m}^{y_m} \cdot \delta_{2m+1}^{z},$$

where δ_{2m+1} must be equal to $+1$ if one of the r_j is odd. The spinor bundle is then

$$\mathbb{S}_\varepsilon = \mathcal{H}^m \times_\varepsilon \Delta_{2m+1},$$

where $g \in \Gamma_r$ acts on the spinor module by multiplication with $\varepsilon(g)$ (which, as an element of the spin group, acts on Δ). The sections of the spinor bundle can thus be identified with equivariant functions:

$$\Gamma(\mathbb{S}_\varepsilon) \simeq \left\{ \varphi \in C^\infty(\mathcal{H}^m, \Delta_{2m+1}) \;\middle|\; \varphi(hg) = \varepsilon(h)\varphi(g) \quad \forall h \in \Gamma_r, g \in \mathcal{H}^1 \right\}.$$

We now want to consider the horizontal Dirac operator on M. In keeping with the notation of [Bär90, AB98] (for $d = 1, T = 2$), we write our basis of $T\mathcal{H}^m$ as $e_{2j-1} = -X_j$, $e_{2j} = -Y_j$ ($j = 1, \ldots, m$) and $e_{2m+1} = \xi$. The spinor connection is given as

$$\nabla^\eta_V \varphi = V(\varphi) + \frac{1}{2} \sum_{j<k} g(\nabla^\eta_V e_j, e_k) e_j e_k \varphi$$

and the horizontal Dirac operator as

$$D^\eta_H \varphi = \sum_{j=1}^{2m} e_j \cdot \nabla^\eta_{e_j} \varphi.$$

Recalling that for a metric connection of torsion T,

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) - T(X; Y, Z) + \tfrac{3}{2}\mathfrak{b}T(X, Y, Z)$$

and that for ∇^η, $T(X; Y, Z) = 0$ if all three arguments are from H, we see that $g(\nabla^\eta_V e_j, e_k) = g(\nabla^g_V e_j, e_k)$ for $V \in \Gamma(H)$ and $j, k = 1, \dots, 2m$. By the results of Bär ([Bär90, p. 36]), we then have that

$$g(\nabla^\eta_{e_j} e_k, e_l) = g(\nabla^g_{e_j} e_k, e_l) = 0 \quad \text{for } j, k, l = 1, \dots, 2m.$$

Furthermore, as ∇^η is adapted,

$$g(\nabla^\eta_{e_j} e_k, \xi) = 0 \quad \text{and} \quad g(\nabla^\eta_{e_j} \xi, \cdot) = 0 \quad \text{for any } j, k = 1, \dots, 2m.$$

Thus, the horizontal Dirac operator may be written as

$$D^\eta_H \varphi = \sum_{j=1}^{2m} e_j \cdot e_j(\varphi).$$

We will now introduce the link between the Dirac operator and representations of $\mathcal{H}^m$, adapting the arguments of [AB98, Bär90] to the horizontal Dirac operator D^η_H. Let R be the right regular representation of $\mathcal{H}^1$ on $L^2(\mathbb{S}_\varepsilon)$, i.e.

$$(R(h)\varphi)(g) = \varphi(gh) \quad \text{for } h, g \in \mathcal{H}^1.$$

We see that the derivative of a smooth spinor in the direction of a left-invariant vector field V on $\mathcal{H}^m$ can be expressed via the right regular representation:

$$V(\varphi)(g) = \frac{d}{dt}\varphi(g \exp(tV))|_{t=0} = (R_*(V)\varphi)(g),$$

where R_* is the derivative of R, i.e. the induced representation on the Lie algebra $\mathfrak{h}^m$. For each $X \in \mathfrak{h}^m$, $R_*(X)$ is an unbounded operator on $L^2(\mathbb{S}_\varepsilon)$ with the space of smooth sections as its domain. We can thus express the horizontal Dirac operator as

$$D^\eta_H \varphi = \sum_{j=1}^{2m} e_j \cdot R_*(e_j)\varphi.$$

The next step will be to decompose $L^2(\mathbb{S}_\varepsilon)$ into subspaces invariant under R and Clifford multiplication and thus, under D^η_H. We will then use the fact that all irreducible representations of $\mathcal{H}^m$ are known and that R will have to decompose into these to explicitly calculate D^η_H and its eigenvalues. We will now introduce the irreducible representations as they are discussed in [AB98, section 2].

In fact, the Heisenberg group admits two families of representations. First, we have a $2m$-parameter family of representations $\pi^\beta\colon \mathcal{H}^m \to U(\mathbb{C})$ ($\beta \in \mathbb{R}^{2m}$) that act via multiplication:

$$\pi^\beta((x,y,z)) = e^{2\pi i\left(\sum_{j=1}^m \beta_{2j-1}x_j + \beta_{2j}y_j\right)}. \tag{3.32}$$

Second, we have a one-parameter family of representations $\pi^\lambda\colon \mathcal{H}^m \to U(L^2(\mathbb{R}^m,\mathbb{C}))$ ($\lambda \in \mathbb{R}\setminus\{0\}$) that act via

$$(\pi^\lambda(x,y,z)u)(t) = e^{2\pi i\lambda(z-\langle t,y\rangle)}u(t-x).$$

Note that π^β acts trivially on the centre $Z(\mathcal{H}^m) = \{(0,0,z)\}$, whereas π^λ acts via multiplication with $e^{2\pi i\lambda z}$ for $(0,0,z) \in Z(\mathcal{H}^m)$. The two representations descend to $\mathfrak{h}^m$ via

$$\pi^\beta_*(X_j) = 2\pi i\beta_{2j-1}\cdot, \quad \pi^\beta_*(Y_j) = 2\pi i\beta_{2j}\,\cdot \quad \text{and} \quad \pi^\beta_*(Z) = 0 \tag{3.33}$$

as well as

$$(\pi^\lambda_*(X_j)u)(t) = -\frac{\partial u}{\partial t_j}(t), \qquad (\pi^\lambda_*(Y_j)u)(t) = -2\pi i\lambda t_j u(t) \text{ and}$$
$$(\pi^\lambda_*(Z)u)(t) = 2\pi i\lambda u(t).$$

We will now discuss the decomposition of $L^2(\mathbb{S}_\varepsilon)$ into subspaces invariant under R and Clifford multiplication and then into copies of the invariant representations π^λ and π^β. As we are using the usual spin structure – unlike in [KU13, Has14] – and the decomposition depends only on $\mathcal{H}^m$ itself (and, in particular, not on the connection ∇^η or the Dirac operator), it is exactly the one discussed in [Bär90, section II.2]. We summarise Bär's results. The decomposition will depend on the spin structure, more precisely on the choice of δ_{2m+1}.

The case $\delta_{2m+1} = +1$. Let the spin structure be fixed and $\delta_{2m+1} = +1$. For a given spinor field $\varphi \in \Gamma(\mathbb{S}_\varepsilon)$, the function $f_{\varphi,g}(t) = (R(0,0,t)\varphi)(g)$ is 1-periodic and can thus be developed in a Fourier series

$$f_{\varphi,g}(t) = \sum_{\alpha\in\mathbb{Z}} \varphi_\alpha(g)\cdot e^{2\pi i\alpha t}$$

Setting $t = 0$ yields a series expansion for φ:

$$\varphi = \sum_{\alpha\in\mathbb{Z}} \varphi_\alpha \tag{3.34}$$

with the additional property that

$$R(0,0,z)\varphi_\alpha = e^{2\pi i\alpha z}\varphi_\alpha. \tag{3.35}$$

Writing this in terms of vector spaces, we have a decomposition

$$L^2(\mathbb{S}_\varepsilon) = H_0 \oplus \bigoplus_{\alpha \neq 0} H_\alpha.$$

By comparing (3.35) with the action of the centre under the irreducible representations, we deduce that each H_α decomposes into copies of the infinite-dimensional representations π^α and H_0 decomposes into (copies of) the representations π^β. A more detailed analysis of the latter decomposition shows that

$$H_0 = \bigoplus_{\beta \in B} H_\beta, \quad B = \left\{ \beta \in (\tfrac{1}{2r}\mathbb{Z} \times \tfrac{1}{2}\mathbb{Z})^m \;\middle|\; e^{2\pi i r \beta_{2j-1}} = \delta_{2j-1}, e^{2\pi i \beta_{2j}} = \delta_{2j} \right\},$$

where each H_β is isomorphic to $\mathbb{C} \times \Delta_{2m+1}$ and R acts via π^β on $\mathbb{C}$ and Clifford multiplication acts on Δ_{2m+1}.

The spaces H_α decomposes into copies of π^α as

$$H_\alpha \simeq \bigoplus_{j=1}^{|\alpha| r_1 \cdots r_m} L^2(\mathbb{R}^m, \Delta_3) = \bigoplus_{j=1}^{|\alpha| r_1 \cdots r_m} L^2(\mathbb{R}^m, \mathbb{C}) \otimes \Delta_3,$$

where on each summand, R acts on the first factor via π^α and Clifford multiplication acts on the second factor. In particular, all spaces $H_\beta, L^2(\mathbb{R}^m, \mathbb{C}) \otimes \Delta$ are stable under both R and Clifford action and thus, under D^η_H.

The case $\delta_{2m+1} = -1$. In the case $\delta_3 = -1$, the function $f_{\varphi,g}$ is 2-periodic. Moreover, every coefficient of the Fourier series of even order is zero, which yields a series expansion

$$\varphi = \sum_{\alpha \in \mathbb{Z}} \varphi_{2\alpha+1}$$

with the property

$$R(0,0,z)\varphi_{2\alpha+1} = e^{2\pi i(\alpha + \frac{1}{2})z} \varphi_{2\alpha+1}.$$

Thus, we obtain a decomposition

$$L^2(\mathbb{S}_\varepsilon) = \bigoplus_{\alpha \in (\mathbb{Z} + \frac{1}{2})} H_\alpha,$$

where each H_α decomposes as before.

This would theoretically allow us to calculate the spectrum of D^η_H in any dimension by calculating the spectra of the restrictions to H_β and H_α. As the dimension grows, this becomes computationally more involved and we restrict ourselves to the cases $m = 1, 2$ here.

The three-dimensional case

We have discussed the action of R_* above and still need to consider the Clifford action. Instead of the description in standard coordinates in (2.1), it will be more convenient to use the basis $u(1), u(-1)$ of $\mathbb{C}^2$ such that matrices U, V, T are given by (as in [AB98])

$$U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then, under the identification $\Delta_3 \simeq \mathbb{C}^2$ via the above basis, the horizontal Dirac operator is given by

$$D_H^\eta \varphi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \cdot R_*(e_1)(\varphi) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot R_*(e_2)(\varphi),$$

where we consider the spinor φ as a function $\varphi \in C^\infty(\mathcal{H}^1, \mathbb{C}^2)$ satisfying the equivariancy condition.

We begin with the case $\delta_3 = 1$. In this case, on each of these spaces H_β, the horizontal Dirac operator has the form

$$\begin{aligned} D_H^\eta|_{H_\beta} &= \pi_*^\beta(e_1) \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \pi_*^\beta(e_2) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i\pi_*^\beta(X) + \pi_*^\beta(Y) \\ -i\pi_*^\beta(X) - \pi_*^\beta(Y) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\pi(\beta_1 + i\beta_2) \\ 2\pi(\beta_1 - i\beta_2) & 0 \end{pmatrix}. \end{aligned}$$

Note that this differs from the full Dirac operator D^g of ∇^g considered in [AB98] only by the zeroes on the diagonal. We deduce from the above form that on H_β, D_H^η has eigenvalues

$$\lambda_\beta^\pm = \pm 2\pi \sqrt{\beta_1^2 + \beta_2^2}.$$

On each of the copies of $L^2(\mathbb{R}, \mathbb{C}) \otimes \mathbb{C}^2$ that form H_α, the horizontal Dirac operator takes the following form:

$$\begin{aligned} D_H^\eta u(t) &= \pi_*^\alpha(e_1) \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} u(t) + \pi_*^\alpha(e_2) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u(t) \\ &= \begin{pmatrix} 0 & -i\pi_*^\alpha(X) + \pi_*^\alpha(Y) \\ -i\pi_*^\alpha(X) - \pi_*^\alpha(Y) & 0 \end{pmatrix} u(t) \\ &= \begin{pmatrix} 0 & i(\frac{d}{dt} - 2\pi\alpha t) \\ i(\frac{d}{dt} + 2\pi\alpha t) & 0 \end{pmatrix} u(t), \end{aligned} \tag{3.36}$$

where $u \in C^\infty(\mathbb{R}, \mathbb{C}^2)$. Again, note that this differs from the "normal" Dirac operator on these spaces only by the zeroes on the diagonal. In order to further analyse this operator, we introduce a special basis of $L^2(\mathbb{R}, \mathbb{C})$, the so-called *Hermite functions* $(h_k)_{k\in\mathbb{N}_0}$ (cf [AB98, section 2]). These satisfy the relations

$$\begin{aligned} h_k'(t) &= th_k(t) + h_{k+1}(t), \\ h_{k+2}(t) &= -2th_{k+1}(t) - 2(k+1)h_k(t). \end{aligned}$$

From these functions, we construct functions $(u_k^\alpha)_{k\in\mathbb{N}_0}$ via

$$u_k^\alpha(t) = h_k(\sqrt{2\pi|\alpha|}\, t).$$

From the above relations for the functions h_k we deduce the following relations for the functions u_k^α:

$$(u_k^\alpha)'(t) = 2\pi|\alpha|t \cdot u_k^\alpha(t) + \sqrt{2\pi|\alpha|} \cdot u_{k+1}^\alpha(t), \tag{3.37}$$

$$u_{k+2}^\alpha = -2\sqrt{2\pi|\alpha|}\, t \cdot u_{k+1}^\alpha(t) - 2(k+1) \cdot u_k^\alpha(t). \tag{3.38}$$

Applying D_H^η in the form (3.36) to these functions, we obtain (for $\alpha > 0$)

$$\begin{aligned} D_H^\eta \begin{pmatrix} u_k^\alpha(t) \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ i((u_k^\alpha)'(t) + 2\pi\alpha t \cdot u_k^\alpha(t)) \end{pmatrix} \\ &\overset{(3.37)}{=} \begin{pmatrix} 0 \\ i(2\pi\alpha t \cdot u_k^\alpha(t) + \sqrt{2\pi\alpha} \cdot u_{k+1}^\alpha(t) + 2\pi\alpha t \cdot u_k^\alpha(t)) \end{pmatrix} \\ &\overset{(3.38)}{=} \begin{pmatrix} 0 \\ i(4\pi\alpha t \cdot u_k^\alpha(t) + \sqrt{2\pi\alpha}(-2\sqrt{2\pi\alpha} \cdot u_k^\alpha(t) - 2u_{k-1}^\alpha(t))) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -2ki\sqrt{2\pi\alpha} \cdot u_{k-1}^\alpha(t) \end{pmatrix}, \\ D_H^\eta \begin{pmatrix} 0 \\ u_{k-1}^\alpha(t) \end{pmatrix} &= \begin{pmatrix} i\sqrt{2\pi\alpha} \cdot u_k^\alpha(t) \\ 0 \end{pmatrix}. \end{aligned}$$

Thus, $(u_k^\alpha(t), 0)^T$ and $(0, u_{k-1}^\alpha(t))^T$ span a D_H^η-invariant subspace for $k \in \mathbb{N}$ and with respect to this basis, D_H^η takes the form

$$\begin{pmatrix} 0 & i\sqrt{2\pi\alpha} \\ -2ki\sqrt{2\pi\alpha} & 0 \end{pmatrix}.$$

Thus, it has the eigenvalues

$$\lambda_\alpha^\pm = \pm 2\sqrt{\pi\alpha k}$$

on this space ($k \in \mathbb{N}$). For $k = 0$, we deduce from the definition of h_k (and u_k^α) that

$$2\pi\alpha t(u_0^\alpha)'(t) + 2\pi\alpha t \cdot u_0^\alpha(t) = 0$$

and thus, $(u_0^\alpha(t),0)^T$ is in the kernel of D_H^η. We are left to consider the case $\alpha<0$. Analogous calculations as above lead to the identities

$$D_H^\eta \begin{pmatrix} u_k^\alpha(t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i\sqrt{2\pi|\alpha|}\cdot u_{k+1}^\alpha \end{pmatrix}, \quad D_H^\eta \begin{pmatrix} 0 \\ u_{k+1}^\alpha(t) \end{pmatrix} = \begin{pmatrix} -2i\sqrt{2\pi|\alpha|}(k+1)u_k^\alpha(t) \\ 0 \end{pmatrix}.$$

Thus, we obtain the eigenvalues

$$\lambda_\alpha^\pm = \pm 2\sqrt{\pi|\alpha|(k+1)}$$

for $k\in\mathbb{N}_0$ and obtain $(0,u_0^\alpha)^T$ as an element of the kernel.

Concerning the case $\delta_3=-1$, we note that the subspace H_0 does not appear and the action of D_H^η on H_α is as before. We summarise the spectral properties of D_H^η, compare also [KU13, section 3.4] and [Has14, Theorem 4.3.4].

Theorem 3.5.1. *Let $\mathcal{H}^1$ be the three-dimensional Heisenberg group and*

$$\Gamma_r = \{\,(rx,y,z)\mid\ x,y,z\in\mathbb{Z}\} < \mathcal{H}^1 \quad (r\in\mathbb{N})$$

be a lattice. On the quotient manifold $H=\Gamma_r\backslash\mathcal{H}^1$, let the spin structure be defined by the homomorphism

$$\varepsilon\colon \Gamma_r\to\mathbb{Z}_2,\quad \varepsilon(rx,y,z)=\delta_1^x\delta_2^y\delta_3^z,\quad \textit{where } \delta_1,\dots,\delta_3\in\{\pm1\}.$$

Then, the eigenvalues of D_H^η are given as follows:

In the case $\delta_3=1$, D_H^η has an infinite-dimensional kernel and the following nonzero eigenvalues:

$$\begin{aligned} \lambda_\beta^\pm &= \pm 2\pi\sqrt{\beta_1^2+\beta_2^2} \quad && (\beta_1,\beta_2)\in\left\{(\beta_1,\beta_2)\in\tfrac{1}{2r}\mathbb{Z}\times\tfrac12\mathbb{Z}\,\middle|\, e^{2\pi i r\beta_1}=\delta_1, e^{2\pi i\beta_2}=\delta_2\right\} \\ \lambda_{\alpha,k}^\pm &= \pm2\sqrt{\pi\alpha k} && \alpha\in\mathbb{N}, k\in\mathbb{N}. \end{aligned}$$

The eigenvalues have the following multiplicities: The multiplicity of $\lambda_\beta^\pm$ has multiplicity 1 for each admissible admissible β and $\lambda_{\alpha,k}^\pm$ has multiplicity $2\alpha r$.

In the case $\delta_3=-1$, which is only possible if r is even, D_H^η has an infinite-dimensional kernel and the following nonzero eigenvalues:

$$\lambda_{\alpha,k}^\pm = \pm2\sqrt{\pi\alpha k} \qquad \alpha\in(\mathbb{N}_0+\tfrac12), k\in\mathbb{N}.$$

which have multiplicity $2\alpha r$.

Remark. Note that the multiplicities of the eigenvlaues $\lambda_\beta^\pm$ can add for multiple β with the same value $\sqrt{\beta_1^2+\beta_2^2}$.

While we have certainly found all eigenvalues, the question whether we have determined the whole spectrum, i.e. whether the other parts of the spectrum beyond the point spectrum are empty, remains open. We have not yet discussed the spectral properties of horizontal Dirac operators in general. Therefore, we will first make more precise what we understand by the spectrum of D^η_H. When talking about the spectrum of an operator, one usually assumes the operator to be a closed operator on a Hilbert space. The horizontal Dirac operator is an unbounded operator $D^\eta_H \colon \Gamma(\mathbb{S}) \subset L^2(\mathbb{S}) \to L^2(\mathbb{S})$. As a differential operator, it is closable, i.e. its closure

$$\overline{D^\eta_H} \colon \operatorname{dom}(\overline{D^\eta_H}) \subset L^2(\mathbb{S}) \to L^2(\mathbb{S}),$$

is well defined, where the domain $\operatorname{dom}(\overline{D^\eta_H})$ is defined to be all spinors $\varphi \in L^2(\mathbb{S})$ for which there exists a spinor $\psi \in L^2(\mathbb{S})$ such that for every sequence $(\varphi_n) \subset \Gamma(\mathbb{S})$ that converges to φ, $(D\varphi_n)$ converges to ψ. As differential operators extend to continuous operators on distributions $D^\eta_H \colon \mathcal{E}'(\mathbb{S}) \to \mathcal{D}'(\mathbb{S})$ (compare appendix A.1 for the definition), it is enough to consider one sequence (φ_n) in the above definition. By the spectrum of D^η_H we now understand the spectrum of $\overline{D^\eta_H}$, i.e. all values $\lambda \in \mathbb{C}$ for which $(\overline{D^\eta_H} - \lambda I)$ does not have an everywhere defined, bounded inverse. Apart from the eigenvalues (also called the point spectrum), there may be other parts of spectrum where $(\overline{D^\eta_H} - \lambda I)$ is injective but still fails to admit a bounded inverse defined on $L^2(\mathbb{S})$.

However, this cannot happen when we have an L^2-basis of eigensections.

Proposition 3.5.2. *Let $E \to M$ be a vector bundle over a compact manifold and $D \colon \Gamma_c(E) \to \Gamma(E)$ a differential operator. Assume there exists an L^2-basis (v_j) of $L^2(E)$ such that $\overline{D}v_j = \lambda_j v_j$. Assume further that the eigenvalues (λ_j) do not accumulate at any finite value. Then, the spectrum of $\overline{D}$ consists only of the eigenvalues (λ_j).*

Proof. We need to show that for any other value $\lambda \in \mathbb{C}$, the operator $\overline{D}_\lambda := \overline{D - \lambda I} = \overline{D} - \lambda I$ admits an everywhere defined bounded inverse $A_\lambda \colon L^2(E) \to L^2(E)$. We define such an inverse: Any $u \in L^2(E)$ can be written as $u = \sum a_j v_j$. Then, we set

$$A_\lambda u = \sum_{j=1}^{\infty} a_j \frac{1}{\lambda_j - \lambda} v_j.$$

As the eigenvalues (λ_j) do not accumulate at λ, there exists a constant $C > 0$ such that $|\frac{1}{\lambda_j - \lambda}| \leq C$. Thus,

$$\|A_\lambda u\|^2_{L^2} = \sum_{j=1}^{\infty} |a_j|^2 \cdot \left|\frac{1}{\lambda_j - \lambda}\right|^2 \cdot \|v_j\|^2_{L^2} \leq C^2 \sum_{j=1}^{\infty} |a_j|^2 \cdot \|v_j\|^2_{L^2} = C^2 \|u\|^2_{L^2},$$

which proves that A_λ is well-defined and bounded. We will now prove that A_λ is indeed an inverse for $\overline{D}_\lambda$. The image of A_λ is in $\operatorname{dom}\overline{D}_\lambda$: We have

$$A_\lambda u = \lim_{n\to\infty}\sum_{j=1}^{n} a_j \frac{1}{\lambda_j-\lambda} v_j,$$

i.e. $A_\lambda u$ is the limit of a sequence in $\operatorname{dom}\overline{D}$. The image of the sequence under $\overline{D}_\lambda$ converges as well, because

$$(\overline{D}-\lambda I)\sum_{j=1}^{n} a_j \frac{1}{\lambda_j-\lambda} v_j = \sum_{j=1}^{n} a_j \frac{1}{\lambda_j-\lambda}(D-\lambda I)v_j = \sum_{j=1}^{n} a_j v_j \overset{n\to\infty}{\longrightarrow} u.$$

Thus, $A_\lambda\varphi$ lies in the domain of the closure of $\overline{D}_\lambda$ which is again $\overline{D}_\lambda$ and $\overline{D}_\lambda A_\lambda u = u$. Finally, for any $u = \sum a_j v_j \in \operatorname{dom}\overline{D}_\lambda$, by definition of $\overline{D}$ and because $(v_j) \in \operatorname{dom}\overline{D}$,

$$\overline{D}_\lambda u = \sum_{j=1}^{\infty} a_j(\lambda_j-\lambda)v_j$$

and the definition of A_λ then implies $A_\lambda\overline{D}_\lambda u = u$. □

Corollary 3.5.3. *The spectrum of the Tanaka-Webster operator D_H^η on the quotients $\Gamma_r\backslash\mathcal{H}^1$ consists only of the eigenvalues listed in Theorem 3.5.1.*

The five-dimensional case

We begin with a discussion of the Clifford action again. Using (2.2), (2.3) and the form of U, V, T above, we deduce

$$\kappa(e_1) = I\otimes U = \begin{pmatrix} 0 & i & & \\ i & 0 & & \\ & & 0 & i \\ & & i & 0 \end{pmatrix}, \qquad \kappa(e_2) = I\otimes V = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix},$$

$$\kappa(e_3) = U\otimes T = \begin{pmatrix} & & -i & 0 \\ & & 0 & i \\ -i & 0 & & \\ 0 & i & & \end{pmatrix}, \qquad \kappa(e_4) = V\otimes T = \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix}.$$

Thus, we deduce from (3.33) that on H_β, D^η_H has the form

$$\begin{aligned}
D^\eta_H|_{H_\beta} &= \sum_{j=1}^{4} -2\pi i\beta_j\kappa(e_j) \\
&= \begin{pmatrix} 0 & 2\pi(\beta_1+i\beta_2) & -2\pi(\beta_3+i\beta_4) & 0 \\ 2\pi(\beta_1-i\beta_2) & 0 & 0 & 2\pi(\beta_3+i\beta_4) \\ -2\pi(\beta_3-i\beta_4) & 0 & 0 & 2\pi(\beta_1+i\beta_2) \\ 0 & 2\pi(\beta_3-i\beta_4) & 2\pi(\beta_1-i\beta_2) & 0 \end{pmatrix}.
\end{aligned}$$

Calculating the eigenvalues of this Hermitian matrix can now be done in the usual way and we obtain

$$\lambda^\pm_\beta = \pm 2\pi\sqrt{\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2},$$

each with multiplicity two.

On the spaces H_α (or, more precisely, on each of the copies of $L^2(\mathbb{R}^m,\mathbb{C})\otimes\Delta_5$), the horizontal Dirac operator has the form

$$\begin{aligned}
D^\eta_H &= \partial_{t_1}\otimes\kappa(e_1) + 2\pi i\lambda t_1\otimes\kappa(e_2) + \partial_{t_2}\otimes\kappa(e_3) + 2\pi i\lambda t_2\otimes\kappa(e_4) \\
&= \begin{pmatrix} 0 & i\partial_{t_1}-2\pi i\alpha t_1 & -(i\partial_{t_2}-2\pi i\alpha t_2) & 0 \\ i\partial_{t_1}+2\pi i\alpha t_1 & 0 & 0 & i\partial_{t_2}-2\pi i\alpha t_2 \\ -(i\partial_{t_2}+2\pi i\alpha t_2) & 0 & 0 & i\partial_{t_1}-2\pi i\alpha t_1 \\ 0 & i\partial_{t_2}+2\pi i\alpha t_2 & i\partial_{t_1}+2\pi i\alpha t_1 & 0 \end{pmatrix}.
\end{aligned}$$

We follow [Bär90] and use Hermite polynomials as a basis of the function space $L^2(\mathbb{R}^2,\mathbb{C})$ again. The Hermite polynomials for the two-dimensional space $\mathbb{R}^2$ have the form

$$h_{j,k}(t_1,t_2) = e^{\frac{\|t\|^2}{2}}\frac{\partial^{j+k}}{\partial t_1^j\partial t_2^k}e^{-\|t\|^2}$$

and satisfy the relations

$$\begin{aligned}
\partial_{t_1}h_{j,k}(t) &= t_1h_{j,k}(t) + h_{j+1,k}(t) && \text{for any } j,k\in\mathbb{N}_0 \\
h_{j+2,k}(t) &= -2t_1h_{j+1,k}(t) - 2(j+1)h_{j,k}(t) && \text{for any } j\in\mathbb{N}_0\cup\{-1\}, k\in\mathbb{N}_0 \\
\partial_{t_2}h_{j,k}(t) &= t_2h_{j,k}(t) + h_{j,k+1}(t) && \text{for any } j,k\in\mathbb{N}_0 \\
h_{j,k+2}(t) &= -2t_2h_{j,k+1}(t) - 2(k+1)h_{j,k}(t) && \text{for any } j\in\mathbb{N}_0 \text{ and } k\in\mathbb{N}_0\cup\{-1\}.
\end{aligned}$$

In the cases $j=-1$, the term $h_{-1,k}$ (which does not exist) does not matter because it has a coefficient zero and analogously for $k=-1$. As in the three-dimensional case, we set

$$u^\alpha_{j,k}(t) = h_{j,k}(\sqrt{2\pi|\alpha|}t)$$

and obtain

$$\partial_{t_1} u^\alpha_{j,k}(t) = 2\pi|\alpha| t_1 \cdot u^\alpha_{j,k}(t) + \sqrt{2\pi|\alpha|} \cdot u^\alpha_{j+1,k}(t) \tag{3.39}$$
$$\partial_{t_2} u^\alpha_{j,k}(t) = 2\pi|\alpha| t_2 \cdot u^\alpha_{j,k}(t) + \sqrt{2\pi|\alpha|} \cdot u^\alpha_{j,k+1}(t)$$
$$u^\alpha_{j+2,k}(t) = -2\sqrt{2\pi|\alpha|}\, t_1 \cdot u^\alpha_{j+1,k}(t) - 2(j+1) u^\alpha_{j,k}(t) \tag{3.40}$$
$$u^\alpha_{j,k+2}(t) = -2\sqrt{2\pi|\alpha|}\, t_2 \cdot u^\alpha_{j,k+1}(t) - 2(k+1) u^\alpha_{j,k}(t)$$

for $j,k \in \mathbb{N}_0$ and

$$u^\alpha_{1,k}(t) = -2\sqrt{2\pi|\alpha|}\, t_1 \cdot u^\alpha_{0,k}, \quad u^\alpha_{j,1}(t) = -2\sqrt{2\pi|\alpha|}\, t_2 \cdot u^\alpha_{j,0} \tag{3.41}$$

for $j,k \in \mathbb{N}_0$. We need to distinguish between the cases $\alpha > 0$ and $\alpha < 0$. For $\alpha > 0$,

$$\begin{aligned}
(i\partial_{t_1} - 2\pi i\alpha t_1) u^\alpha_{j,k}(t) &\overset{(3.39)}{=} i\left(2\pi\alpha t_1 \cdot u^\alpha_{j,k}(t) + \sqrt{2\pi\alpha} \cdot u^\alpha_{j+1,k}(t) - 2\pi\alpha t_1 \cdot u^\alpha_{j,k}(t)\right) \\
&= i\sqrt{2\pi\alpha} \cdot u^\alpha_{j+1,k}, \\
(i\partial_{t_1} + 2\pi i\alpha t_1) u^\alpha_{j,k}(t) &\overset{(3.39)}{=} i\left(2\pi\alpha t_1 \cdot u^\alpha_{j,k}(t) + \sqrt{2\pi\alpha} \cdot u^\alpha_{j+1,k}(t) + 2\pi\alpha t_1 \cdot u^\alpha_{j,k}(t)\right) \\
&\overset{(3.40)}{=} i\left(4\pi\alpha t_1 \cdot u^\alpha_{j,k}(t) + \sqrt{2\pi\alpha}\left(-2\sqrt{2\pi\alpha}\, t_1 \cdot u^\alpha_{j,k} - 2j \cdot u^\alpha_{j-1,k}\right)\right) \\
&= -i2j\sqrt{2\pi\alpha} \cdot u^\alpha_{j-1,k}.
\end{aligned}$$

We have analogous relations if we replace t_1 by t_2 and thus obtain that

$$(u^\alpha_{j,k}, 0,0,0), (0, u^\alpha_{j-1,k}, 0, 0), (0,0,u^\alpha_{j,k-1}, 0), (0,0,0,u^\alpha_{j-1,k-1})$$

form a D^η_H-invariant subspace for each pair $(j,k) \in \mathbb{N}^2$. Restricted to this subspace, the horizontal Dirac operator has matrix form (with respect to this basis)

$$D^\eta_H \sim \begin{pmatrix} 0 & i\sqrt{2\pi\alpha} & -i\sqrt{2\pi\alpha} & 0 \\ -i2j\sqrt{2\pi\alpha} & 0 & 0 & i\sqrt{2\pi\alpha} \\ i2k\sqrt{2\pi\alpha} & 0 & 0 & i\sqrt{2\pi\alpha} \\ 0 & -i2k\sqrt{2\pi\alpha} & -i2j\sqrt{2\pi\alpha} & 0 \end{pmatrix}.$$

This matrix has eigenvalues

$$\lambda^\pm_{\alpha,j,k} = 2\sqrt{\pi\alpha(j+k)},$$

each wih multiplcity 2. Furthermore, we deduce from (3.41) that

$$(u^\alpha_{0,k}, 0,0,0), (0,0,u^\alpha_{0,k-1}, 0)$$

for $k \in \mathbb{N}$ as well as

$$(u^\alpha_{j,0}, 0,0,0), (0, u^\alpha_{j-1,0}, 0, 0)$$

form D^η_H-invariant subspaces. These yield the eigenvalues $\lambda^\pm_{\alpha,0,k}$ and $\lambda^\pm_{\alpha,j,0}$ with definition as above for $j,k\in\mathbb{N}$. Finally, again from (3.41), we deduce that $(u^\alpha_{0,0},0,0,0)$ is an element of the kernel.

For $\alpha<0$, we have

$$(i\partial_{t_1}-2\pi i\alpha t_1)u^\alpha_{j,k}=-i2j\sqrt{2\pi|\alpha|}\cdot u^\alpha_{j-1,k},$$
$$(i\partial_{t_1}+2\pi i\alpha t_1)u^\alpha_{j,k}=-i\sqrt{2\pi|\alpha|}\cdot u^\alpha_{j+1,k}.$$

Thus, a D^η_H-invariant subspace is spanned by

$$(u^\alpha_{j-1,k-1},0,0,0),(0,u^\alpha_{j,k-1},0,0),(0,0,u^\alpha_{j-1,k},0),(0,0,0,u^\alpha_{j,k})$$

Further invariant spaces are spanned by

$$(0,0,u^\alpha_{j-1,0},0),(0,0,0,u^\alpha_{j,0})\quad\text{as well as}\quad(0,u^\alpha_{0,k-1},0,0),(0,0,0,u^\alpha_{0,k}).$$

Writing D^η_H in matrix form with respect to each of these bases, we can calculate the eigenvalues and again obtain eigenvalues $\lambda^\pm_{|\alpha|,j,k}$ for $j,k\in\mathbb{N}_0$. Note that for $j=k=0$ this gives an element of the kernel.

Finally, we need to consider the case $\delta_{2m+1}=-1$. The eigenspaces H_β do not appear here and the H_α appear for different indices α, but the calculations for D^η_H restricted to one of these spaces remain the same. We can summarize the results in the following theorem, compare also [Has14, Theorem 4.3.4].

Theorem 3.5.4. *Let $\mathcal{H}^2$ be the five-dimensional Heisenberg group and*

$$\Gamma_r=\left\{(r_1x_1,r_2x_2,y,z)\,\middle|\,(x,y,z)\in\mathbb{Z}^5\right\}<\mathcal{H}^2,$$

where $r_1,r_2\in\mathbb{N}$ such that r_1 divides r_2, be a lattice. On the quotient manifold $H=\Gamma_r\backslash\mathcal{H}^2$, let the spin structure be defined by the homomorphism

$$\varepsilon\colon\Gamma_r\to\mathbb{Z}_2,\qquad\varepsilon(rx,y,z)=\delta_1^{x_1}\delta_2^{y_1}\delta_3^{x_2}\delta_4^{y_2}\delta_5^{z},$$

where $\delta_1,\dots,\delta_4\in\{\pm1\}$ and $\delta_{2m+1}=1$ if one r_j is odd and $\delta_5\in\{\pm1\}$ otherwise. Then, the spectrum of D^η_H consists only of the following eigenvalues:

In the case $\delta_3=1$, D^η_H has an infinite-dimensional kernel and the following nonzero eigenvalues:

$$\lambda^\pm_\beta=\pm2\pi\sqrt{\beta_1^2+\cdots+\beta_4^2}\qquad\beta\in B,$$
$$\lambda^\pm_{\alpha,j,k}=\pm2\sqrt{\pi\alpha(j+k)}\qquad\alpha\in\mathbb{N},j,k\in\mathbb{N}_0,$$

where

$$B=\left\{(\beta_1,\dots,\beta_4)\in(\tfrac{1}{2r}\mathbb{Z}\times\tfrac{1}{2}\mathbb{Z})^2\,\middle|\,e^{2\pi ir\beta_{2\mu-1}}=\delta_{2\mu-1},e^{2\pi i\beta_{2\mu}}=\delta_{2\mu}\ (\mu=1,2)\right\}.$$

The eigenvalues have the following multiplicities: The value $\lambda_\beta^\pm$ *has multiplicity 2 for every* β *and* $\lambda_{\alpha,j,k}^\pm$ *has multiplicity* $4\alpha r_1 \cdot r_2$ *for each pair* (j,k).

In the case $\delta_3 = -1$, D_H^η *has an infinite-dimensional kernel and the following nonzero eigenvalues:*

$$\lambda_{\alpha,j,k}^\pm = \pm 2\sqrt{\pi\alpha k} \qquad\qquad \alpha \in (\mathbb{N}_0 + \tfrac{1}{2}), k \in \mathbb{N}.$$

which have multiplicity $4\alpha r_1 r_2$ *for each pair* (j,k).

Proof. That these eigenvalues form the entire spectrum is a consequence of Proposition 3.5.2. Alternatively, one could deduce this from the general theory in the following chapter. □

4 Analysis of the horizontal Dirac Operators: Heisenberg Calculus

4.1 Introduction

The Heisenberg Calculus is a replacement for the classic symbolic calculus for (pseudo)differential operators. Its aim is to provide a replacement for ellipticity that still guarantees the “nice” analytic and spectral properties that hold for an elliptic operator $P\colon \Gamma_c(E) \to \Gamma(E)$, including the following:

- Every elliptic differential operator is *hypoelliptic*, i.e. if we extend P to distributions, we have that if Pu is smooth, so is u.
- Every elliptic differential operator admits a *parametrix*, i.e. a continuous linear operator $Q\colon \Gamma_c(E) \to \Gamma(E)$ such that $QP = Id + S_1$ and $PQ = Id + S_2$, where S_1, S_2 are smoothing operators, i.e. $S_j\colon \Gamma_c(E) \to \Gamma(E)$ extend to continuous linear operators $S_j\colon \mathcal{E}'(E) \to \mathcal{E}(E)$, compare Appendix A.1 for the definition of $\mathcal{E}'(E)$.
- On a closed (i.e. compact and without boundary) manifold, every formally self-adjoint elliptic operator has discrete pure point spectrum.

We want to study whether these properties hold for operators that, roughly speaking, look like an elliptic operator in the direction of a codimension 1 subbundle $H \subset TM$ and only have contributions of lower order in the remaining direction. An example of operators that we have in mind are the horizontal Dirac operators, which look like a normal (elliptic) Dirac operator in the H-direction but do not have derivatives in a transversal direction. More generally, let M^{d+1} be a manifold equipped with a codimension 1 subbundle $H \subset TM$ that is not necessarily integrable. The following two observations may motivate the approach taken by the Heisenberg calculus:

- Let $X_1, ..., X_d$ be a pointwise basis of H, X_0 be transversal to H and consider the differential operator P (locally) given by

$$P = -\sum_{j=1}^{d} X_j^2 + \mu(x) X_0.$$

Then, the study of certain operators of this type has shown that whether P is hypoelliptic depends on μ. In other words, when the operator is not elliptic, the hypoellipticity will depend on contributions of lower order in the transversal direction. If we want a symbol of operators that is able to tell us whether the operator is hypoelliptic, it will need to reflect these contributions of lower order.

- If H is not integrable (which is, in particular, the case for a contact or CR manifold), the commutator of $X_1, X_2 \in \Gamma(H)$ will not necessarily be in H. Thus, the product of two operators that derive only in direction of H may have derivatives in a transversal direction. Moreover, by the previous point, this contribution in the transversal direction may be crucial for the analytic properties of the operator. This also means that, because the symbols should reflect these contributions, the commutator does not vanish when going to symbols and the calculus will therefore in general be noncommutative.

The Heisenberg calculus takes this into account by considering derivatives in directions transversal to H as second-order operators and producing a symbol algebra that takes into account the second observation. The principal symbol of an operator P then defines a *model operator* at each point that is a left-invariant (pseudo)differential operator on a Lie group. For such operators, existence of a parametrix and hypoellipticity are determined via the representation-theoretic calculus of Rockland operators and hypoellipticity of the model operator at each point will imply hypoellipticity of P.

With the application of sub-Dirac operators in mind, we will sometimes restrict our discussion from general manifolds with a codimension-one subbundle to strictly pseudoconvex CR manifolds when it simplifies notation or calculations. While sub-Dirac operators are differential operators, we will need to consider pseudodifferential operators as well, as these provide parametrices and partial inverses for differential operators.

How to read this chapter

We have tried to make this discussion as accessible as possible to differential geometers without much previous understanding of functional analysis. At the same time, we tried to still accurately present the theory of Heisenberg calculus (although we will usually refer to the original papers for the proofs). We have tried to split the necessary knowledge and the more technical details into different sections.

The reader who wants to know only the analytical properties of the horizontal Dirac operators will find them in Proposition 4.5.16 and Theorem 4.5.19. The reader who wants to gain some insight into the underlying structures and results should read the definition of the tangent group in section 4.2 (and skip the discussion of adapted coordinates), section 4.3 on differential operators in the Heisenberg calculus and then continue with section 4.5 on the hypoellipticity properties of these operators. In that case, one should simply read the class Ψ_H of Heisenberg pseudodifferential operators as an appropriate class of linear operators containing the inverses of sorts of H-elliptic differential operators. Finally, if one wants to know more about these pseudodifferential operators, in particular how they are constructed and why they give rise to a useful symbol calculus, section 4.4 provides the details.

4.2 Heisenberg manifolds and their tangent groups

In this section, we introduce *Heisenberg manifolds*. These are the most general type of manifold on which we can develop the Heisenberg calculus and contain, among other, metric contact and CR manifolds. We will discuss the appropriate tangent spaces, which turn out to be groups, and adapted coordinates on these manifolds. The discussion mostly follows [Pon08, section 2.1]

We begin by introducing the type of manifold on which we will develop the Heisenberg calculus.

Definition. A *Heisenberg manifold* is a smooth manifold M of dimension $d+1$ together with a codimension 1 distribution $H \subset TM$. An *H-frame* on $U \subset M$ is a tuple of vector fields $X_0, X_1, ..., X_d \in \mathfrak{X}(U)$ such that X_0 is transversal to H and $(X_1, \ldots, X_d)$ are a pointwise basis of H.

The appropriate notion of morphism for this structure is the following.

Definition. Let (M, H) and $(\hat{M}, \hat{H})$ be Heisenberg manifolds. A *Heisenberg isomorphism* is a diffeomorphism $\Phi\colon M \to \hat{M}$ such that $d\Phi(H) = \hat{H}$.

We will associate to each point $a \in M$ a two-step nilpotent tangent Lie group $T_H M_a$. This group will reflect the difference between vectors in H and transversal vectors. As indicated in the intropduction, this difference is important in the Heisenberg calculus.

To begin with, we note the following property of the commutator of vector fields:

$$[fX, gY]_{\mathfrak{X}(M)} = fg[X,Y]_{\mathfrak{X}(M)} + fX(g)Y - gY(f)X \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

This implies the following lemma.

Lemma 4.2.1 ([Pon08, Lemma 2.1.3]). *The commutator $[\cdot,\cdot]_{\mathfrak{X}(M)}$ of vector fields on M induces a TM/H-valued 2-form $\mathcal{L}\colon H \times H \to TM/H$ such that for any $X, Y \in \Gamma(H)$*

$$\mathcal{L}_a(X(a), Y(a)) = \left[[X,Y]_{\mathfrak{X}(M)}(a)\right]_{H_a},$$

where $[\cdot]_{H_a}$ denotes a class in TM_a/H_a.

Definition. The 2-form $\mathcal{L}$ is called the *Heisenberg-Levi form* of M.

We used the term Heisenberg-Levi form instead of Levi form (as it is usually called in the literature) here to distinguish it from the Levi form of a CR manifold $L_\eta(X,Y) = \frac{1}{2}d\eta(X,JY)$.

Abstractly, we can now define the tangent group $T_H M_a$ as follows: It is the simply-connected nilpotent group associated with the following Lie algebra: Let $\mathfrak{t}_H\mathfrak{m}_a$ be $H_a \oplus (TM_a/H_a)$ as a vector space, equipped with the following Lie algebra structure:

$$\begin{aligned} [X,Y] &= \mathcal{L}_a(X,Y) && \text{for all } X,Y \in H_a, \\ [\mathfrak{t}_H\mathfrak{m}_a, Z] &= 0 && \text{for all } Z \in TM_a/H_a, \end{aligned}$$

It can easily be checked that this is a 2-step nilpotent or abelian Lie algebra. Any Lie algebra is the Lie algebra of a Lie group. For nilpotent and abelian algebras, this relationship is particularly easy.

Proposition 4.2.2 ([Kna02, theorem 1.127])**.** *Let G be a simply connected Lie group, $\mathfrak{g}$ its Lie algebra and let $\mathfrak{g}$ be nilpotent. Then, the exponential map* $\exp\colon \mathfrak{g} \to G$ *is a diffeomorphism.*

This means, in particular, that for a nilpotent Lie algebra, we can choose the group to have the same underlying space (which, as a vector space, is simply connected) and the exponential map as the identity. One can then use the Baker-Campbell-Hausdorff formula (written down for a 2-step nilpotent group, for an abelian group the commutator term vanishes)

$$\exp(X)\exp(Y) = \exp\left(X + Y + \tfrac{1}{2}[X,Y]\right)$$

to determine the group structure.

We now apply this for the tangent Lie group $T_H M_a$. Its underlying space is $H_a \oplus T_H M_a/H_a$ and the product is defined by

$$(X + X_0)\cdot(Y + Y_0) = (X+Y) + (X_0 + Y_0 + \tfrac{1}{2}\mathcal{L}_a(X,Y)) \tag{4.1}$$

for $X,Y \in H_a$ and $X_0, Y_0 \in TM_a/H_a$. There is another way to construct $T_H M_a$ that uses equivalence classes of curves on M through a. This construction can be found in [vE10].

As one might expect, a Heisenberg isomorphism induces a group morphism of the tangent group.

Lemma 4.2.3 ([Pon08, Proposition 2.1.8])**.** *Let $\Phi\colon (M,H) \to (\hat{M},\hat{H})$ be a Heisenberg isomorphism. The differential $d\Phi$ descends to an isomorphism of vector bundles $\bar{d\Phi}\colon TM/H \to T\hat{M}/\hat{H}$. Choose a vector X_0 transversal to H_a and define $D_H\Phi_a\colon T_H M_a \to T_{\hat{H}}\hat{M}_{\Phi(a)}$ by*

$$D_H\Phi_a([X_0]_{H_a} + X) = \bar{d\Phi}([X_0]_{\hat{H}_a}) + d\Phi(X)$$

for $X \in H_a$. Then, $D_H\Phi_a$ is independent of the choice of X_0 and a group isomorphism.

Definition. The group isomorphism $D_H\Phi_a\colon T_HM_a \to T_{\hat{H}}\hat{M}_{\Phi(a)}$ is called the *Heisenberg differential* of Φ.

Let us consider the case of a strictly pseudoconvex CR manifold (M^{2m+1}, H, J, η), equipped with its Webster metric $g_\eta = L_\eta + \eta \otimes \eta$. In particular, $H \subset TM$ is a codimension one subbundle that is the kernel of η and η satisfies that $\eta \wedge (d\eta)^m$ vanishes nowhere and the normal bundle $N = TM/H$ is trivialised by the Reeb vector field ξ. In this case, one can consider *H-frames* of the form where $X_0 = \xi$. The trivialisation of the quotient TM/H will make notation a little more convenient at times, but the calculus does not change in any significant way.

For a CR manifold, we can then interpret the Heisenberg-Levi form as a real-valued form $\mathcal{L}\colon H \times H \to \mathbb{R}$ given by

$$\mathcal{L}(X,Y) = g([X,Y]_{\mathfrak{X}(M)}, \xi),$$

write the tangent algebra as $\mathfrak{t}_H\mathfrak{m}_a = H_a \oplus \mathbb{R}\xi(a)$ and write the algebra structure as $[X,Y] = \mathcal{L}_a(X,Y)\xi(a)$ for $X, Y \in H_a$.

We come back to the case of a general Heisenberg manifold. Given an H-frame, we can introduce *standard coordinates* on $\mathfrak{t}_H\mathfrak{m}_a$ and T_HM_a via

$$x \mapsto x_0X_0(a) + x_1X_1(a) + \cdots + x_dX_d(a). \tag{4.2}$$

In these coordinates, the product (4.1) on T_HM_a is given by

$$x \cdot y = (x_0 + y_0 + \frac{1}{2}\sum_{j,k=1}^{d} L_{jk}(a)x_jy_k, x_1 + y_1, \ldots, x_d + y_d), \tag{4.3}$$

where the coefficients L_{jk} are implicitly given by the Levi form:

$$\mathcal{L}(X_j, X_k) = L_{jk}[X_0]_{H_a}. \tag{4.4}$$

As a Lie algebra, $\mathfrak{t}_H\mathfrak{m}_a$ can be described as the algebra of left-invariant vector fields on T_HM_a. In this particular situation, we can use left-invariant vector fields that are induced by from vector fields on M: Let $(X_0, \ldots X_d)$ be an H-frame on $U \subset M$ around a. Then, by definition of $\mathfrak{t}_H\mathfrak{m}_a$, $X_j(a) \in \mathfrak{t}_H\mathfrak{m}_a$ and we can define a left-invariant vector field via

$$X_j^a f(x) = \frac{d}{dt} f(x \cdot \exp(tX_j(a)))|_{t=0}.$$

In the coordinates (4.2), we have [Pon08, (2.1.14)]

$$X_0^a(x) = \frac{\partial}{\partial x_0}(x) \quad \text{and} \quad X_j^a(x) = \frac{\partial}{\partial x_j}(x) - \frac{1}{2}\sum_{k=0}^{d} L_{jk}(a)x_k\frac{\partial}{\partial x_0}(x). \tag{4.5}$$

Structure of the tangent group

We go back to the case of a general Heisenberg manifold in this section. The structure of the Heisenberg manifold is reflected in the structure of the tangent group $T_H M_a$, or more precisely, how closely it resembles the Heisenberg group $\mathcal{H}^n$. Recall that this group is topologically $\mathbb{R}^{2n+1}$ with the group structure (1.2).

Proposition 4.2.4 ([Pon06, Prop 2.8])**.** *Let (M^{d+1}, H) be a Heisenberg manifold, $a \in M$. The Levi form $\mathcal{L}_a$ has rank $2k$ if and only if the tangent group $T_H M_a \simeq \mathcal{H}^k \times \mathbb{R}^{d-2k}$ as a graded Lie group.*

The Levi form $\mathcal{L}$ has constant rank $2k$ if and only if the tangent bundle $T_H M = \coprod_{a \in M} T_H M_a$ is a fibre bundle with fibre $\mathcal{H}^k \times \mathbb{R}^{d-2k}$.

The two "extremes" of this spectrum of possible tangent group structures are foliations and contact manifolds.

Proposition 4.2.5 ([Pon06, Prop 2.9, 2.10])**.** *Let (M^{d+1}, H) be a Heisenberg manifold.*

1. *The following statements are equivalent:*
   - *(i) (M, H) is a foliation.*
   - *(ii) The Heisenberg-Levi form $\mathcal{L}$ vanishes*
   - *(iii) As a Lie group bundle, $T_H M$ is isomorphic to the abelian bundle $H \oplus TM/H$.*
2. *The following statements are equivalent:*
   - *(i) (M, H) is a contact manifold*
   - *(ii) The Heisenberg-Levi form is nowhere degenerate*
   - *(iii) The tangent bundle $T_H M$ is a Lie group bundle with fibre $\mathcal{H}^{d/2}$.*

4.2.1 Coordinates adapted to Heisenberg manifolds

We would like to introduce two sets of coordinates on M that are in some way adapted to the Heisenberg structure on M. Fix a point $a \in M$. Given some local chart (U, φ) with $\varphi(a) = 0$ and an H-frame $(X_0, \ldots, X_{2m})$, we denote Y_j the push-forward of X_j under φ. There is always a linear change of coordinates such that the new coordinates $\psi_a = (y_0, \ldots, y_{2m})$ satisfy $\psi_a(a) = 0$ and $X_j(a) = \frac{\partial}{\partial y_j}(a)$. In fact, this coordinate change can be constructed as follows (cf [Pon08, (2.1.16)]): Let $Y_j(x) = \sum_{k=0}^d b_{jk}(x) e_k$, where e_k is the k-th vector of the standard basis of $\mathbb{R}^{d+1}$. Then, the coordinate change is given by

$$(B(0)^T)^{-1} x, \text{ where } B = (b_{jk}). \tag{4.6}$$

Definition. We will call the coordinates ψ_a *H-adapted coordinates at a.*

These coordinates are called *privileged coordinates* in [Pon08] and *a-coordinates* in [BG88].

We will next provide another set of coordinates that come from another construction of $\mathfrak{t}_H\mathfrak{m}$. These new coordinates will be a useful technical tool at some points and will show that our tangent group structure is equivalent to the y-group structure of [BG88, section 11]. With the setting and notation as above, in H-adapted coordinates surjective on $\mathbb{R}^{d+1}$, we have $Y_j(0) = e_j$ and thus

$$Y_j(y) = e_j + \sum_{k=0}^{d} a_{jk}(y)e_k \tag{4.7}$$

with some smooth functions a_{jk} with $a_{jk}(0) = 0$.

We introduce the dilation

$$\delta_t(x_0, \dots, x_d) = (t^2x_0, ty_1, \dots, tx_d) \quad (t > 0)$$

on $\mathbb{R}^{d+1}$ and extend it to functions f and operators on functions P (including vector fields) as follows:

$$\delta_t f(x) = f(\delta_t x) \quad \text{and} \quad \delta_t^* P = \delta_t^{-1} \circ P \circ \delta_t = \delta_{1/t} \circ P \circ \delta_t.$$

We call a vector field $X \in \mathfrak{X}(\mathbb{R}^{d+1})$ *homogeneous of degree k* if $\delta_t^* X = t^k X$. Obviously, the canonical basis field e_0 is homogeneous of degree 2 in this sense and the other canonical fields $e_1, \dots, e_d$ are homogeneous of degree 1.

Remark. Note that in contrast to the work of Ponge [Pon08, section 2.1.2], the vector fields e_k are homogeneous of positive degree. Because of this difference, we will give detailed calculations in the remainder of this section.

We now define vector fields on $\mathbb{R}^{d+1}$ that capture the homogeneous parts of the Y_j:

$$\begin{aligned} X_0^{(a)} &= \lim_{t\to\infty} t^{-2}\delta_t^* Y_0 \\ X_j^{(a)} &= \lim_{t\to\infty} t^{-1}\delta_t^* Y_j \quad (j = 1, ..., d) \end{aligned}$$

Lemma 4.2.6. *The vector fields $X_j^{(a)}$ are well-defined and given by the following formulae, where $A_{jk} = \frac{\partial a_{j0}}{\partial x_k}(0)$:*

$$\begin{aligned} X_0^{(a)}(y) &= e_0 \\ X_j^{(a)}(y) &= e_j + \sum_{k=1}^{d} A_{jk} \cdot y_k \cdot e_0 \qquad (j = 1, \dots, d) \end{aligned}$$

Proof. Proving the formulae will yield the existence. Using the homogeneity of the basis fields and (4.7), we have

$$t^{-2}\delta_t^* Y_0 = t^{-2}\left(t^2(1+\delta_{1/t}a_{00})e_0 + \sum_{k=1}^{d} t\delta_{1/t}a_{0k}e_k\right).$$

As $a_{jk}(0)=0$ and the a_{jk} are continuous, $\lim_{t\to\infty} t^{-2}t\delta_{1/t}a_{0k}e_k = 0$. This yields the first formula. Furthermore, using the smoothness of a_{jk}, we obtain that

$$\lim_{t\to\infty} t^{-1}t^2\delta_{1/t}a_{j0}(y) = \sum_{k=1}^{d}\frac{\partial a_{j0}}{\partial y_k}(0)y_k$$

and this, together with a similar argument as above for the other terms, yields the second formula. □

One can show that the vector fields $X_j^{(a)}$ are homogeneous of degrees 2 ($j=0$) and 1 ($j=1,\dots,d$). These vector fields span a Lie algebra and we calculate its structure:

$$\begin{aligned}
[X_j^{(a)}, X_0^{(a)}](y) &= \left[e_j + \sum_{k=1}^{d} A_{jk}\cdot y_k\cdot e_0, e_0\right]\\
&= \sum_{k=1}^{d}[A_{jk}\cdot y_k\cdot e_0, e_0]\\
&= -\sum_{k=1}^{d} A_{jk}\frac{\partial}{\partial y_0}(y_k)e_0 = 0.
\end{aligned}$$

Similarly,

$$[X_j^{(a)}, X_l^{(a)}] = (A_{lj}-A_{jl})X_0^{(a)}.$$

Therefore, the vector fields $(X_j^{(a)})$ span a 2-step nilpotent (or abelian) Lie algebra.

Definition. The Lie algebra spanned by the vector fields $(X_j^{(a)})$ is denoted $\mathfrak{g}^{(a)}$. The associated simply connected Lie group is denoted $G^{(a)}$.

These vector fields are exactly the left-invariant extensions of $e_j(0)$ for the group law

$$x\cdot y = \left(x_0 + y_0 + \sum_{k=1}^{d} A_{kj}x_jy_k, x_1+y_1+\dots x_d+y_d\right),$$

i.e. $\mathfrak{g}^{(a)}$ is the Lie algebra of $\mathbb{R}^{d+1}$ with that group law. We will now compare this Lie algebra with $\mathfrak{t}_H\mathfrak{m}_a$. Let L_{jk} be the coefficients of $\mathcal{L}$ near a as in (4.4). Because

$[X,Y](f) = XY(f) - YX(f)$, one easily sees that $[\delta_t^* X, \delta_t^* Y] = \delta_t^*[X,Y]$. Thus, for all $j,k = 1,\dots,d$,

$$\begin{aligned}\left[X_j^{(a)}, X_k^{(a)}\right] &= \left[\lim_{t\to\infty} t^{-1}\delta_t^* Y_j, \lim_{t\to\infty} t^{-1}\delta_t^* Y_k\right] \\ &= \lim_{t\to\infty} t^{-2}\delta_t^*[Y_j, Y_k] \\ &= \lim_{t\to\infty} t^{-2}\delta_t^* L_{jk} Y_0 = L_{jk} X_0^{(a)}.\end{aligned}$$

Therefore, $\mathfrak{g}^{(a)} \simeq \mathfrak{t}_H\mathfrak{m}_a$ and thus $G^{(a)} \simeq T_H M_a$. Moreover, $L_{jk} = A_{kj} - A_{jk}$. Using this and the explicit multiplication structures for $G^{(a)}$ above and for $T_H M_a$ in (4.3) (in the standard coordinates (4.2)), one may check that $\phi_a \colon G^{(a)} \to T_H M_a$ as follows (still in coordinates) is an isomorphism.

$$(x_0, \dots, x_d) \longmapsto \left(x_0 + \frac{1}{4}\sum_{k=1}^{d}(A_{kj} - A_{jk})x_k\right). \tag{4.8}$$

One may then show (cf [Pon06, Lemma 1.17]) that $(d\phi_a)(X_j^{(a)}) = X_j^a$.

Definition. The coordinates provided by $\varepsilon_a = \phi_a \circ \psi_a$ (recall that ψ_a are the adapted coordinates) are called *Heisenberg coordinates* at a with respect to the H-frame $(X_0, \dots, X_d)$.

Just like the exponential map on Riemannian manifolds, the Heisenberg coordinates provide coordinates $\varepsilon_a \colon U \to T_H M_a$ (with standard coordinates for $T_H M_a$) on the tangent space. However, unlike the normal coordinates, the Heisenberg coordinates are not unique. We will therefore close this section by giving two applications of this constructions and the Heisenberg coordinates that will show that they are indeed very useful.

At a, the image of X_j under $d(\varepsilon_a)_a$ is X_j^a, which in turn is the image of the homogeneous vector field $X_j^{(a)}$ under the group isomorphism ϕ_a. Thus, the above discussion allows us to conclude that the left-invariant vector fields X_j^a capture the homogeneous part of X_j in the following sense.

Lemma 4.2.7 (cf [Pon08, Formula (2.1.28)])**.** *Let (M,H) be a Heisenberg manifold, $(X_0, \dots, X_d)$ an H-frame and ε_a the associated Heisenberg coordinates around a. Then,*

$$\begin{aligned}&\delta_t^*(d(\varepsilon_a)_a X_0) = t^2 X_j^a + O(1), \\ &\delta_t^*(d(\varepsilon_a)_a X_j) = t X_j^a + O(t^{-1}) \quad \textit{for } j = 1, \dots, d.\end{aligned}$$

As a second application, in Heisenberg coordinates one can see that the Heisenberg differential is actually the approximation of first order of a Heisenberg isomorphism.

Proposition 4.2.8 ([Pon06, Proposition 2.21])**.** *Let $\Phi\colon (M,H) \to (\hat{M},\hat{H})$ be a Heisenberg isomorphism. Choose Heisenberg coordinates at a and $\Phi(a)$ and denote $\tilde{\Phi}$ the Heisenberg isomorphism expressed in these coordinates. Then, near $x = 0$ zero, we have*

$$\tilde{\Phi}(x) = D_H\tilde{\Phi}_0(x) + (O(\|x\|^3), O(\|x\|^2), \dots, O(\|x\|^2)),$$

where $\|x\|^2 = (x_0^2 + (x_1^2 + \cdots + x_{2m}^2)^2)^{1/4}$.

This property will be important for proving that the pseudodifferential operators we are about to introduce are invariant under an appropriate change of coordinates, see Proposition 4.4.18.

4.3 Heisenberg order and Heisenberg symbol of differential operators

We now turn to operators. We first consider only differential operators and define their Heisenberg order and (principal) symbol. The structure of the symbol algebra will serve as motivation for the choices we make when defining Heisenberg pseudodifferential operators. The interest of the pseudodifferential operators is that they will provide parametrices and partial inverses for differential operators. This section is largely based on chapter 3 of [vE05].

The main change in the Heisenberg calculus for differential operators is the notion of order of an operator. Any other changes (different symbol map, different interpretation of the symbol, noncommutativity of the symbol calculus) essentially follow from this change. Throughout this section, let $E \to M$ be a vector bundle of fibre $\mathbb{C}^r$ over a Heisenberg manifold. We will us multi-index-notation, i.e. for $\gamma = (\gamma_0, ..., \gamma_d)$, we write $X^\gamma = X_0^{\gamma_0} \cdots X_d^{\gamma_d}$ and $\langle\gamma\rangle = 2\gamma_0 + \gamma_1 + \cdots + \gamma_d$. Note the factor 2 in front of γ_0 in $\langle\gamma\rangle$, which thus differs from the usual norm (or absolute value) $|\gamma|$ of a multiindex.

Definition. We say that a linear operator $P\colon \Gamma(E) \to \Gamma(E)$ is a *differential operator of Heisenberg order k* if in any local trivialisation (U,ϕ) of E, and for any H-frame $X_0, X_1, ..., X_d$ on U, there exist local sections $b_\gamma \in C^\infty(U, \mathbb{C}^{r\times r})$ such that P has the form

$$\phi \circ P \circ \phi^{-1} = \sum_{\langle\gamma\rangle \leq k} b_\gamma X^\gamma, \tag{4.9}$$

where we extended ϕ to local sections.

Note that the factor 2 in front of γ_0 in $\langle\gamma\rangle$ implies that we consider derivatives in X_0-direction as second-order operators!

Remark. We may alternatively describe the algebra of differential operators of Heisenberg order k as follows: The order of a vector field $ord(X)$ is 1 if $X \in \Gamma(H)$ and 2 otherwise. Then, operators of Heisenberg order k are spanned by all operators of the following type:

$$b \cdot \nabla_{Y_1} \cdots \nabla_{Y_l} \quad \text{where } Y_1, ..., Y_l \in \mathfrak{X}(M) \colon \sum_{j=1}^{l} ord(Y_j) \leq k,$$

where b is a field of endomorphisms of E and ∇ is any covariant derivative on E. This is equivalent to the above definition because in a local trivialisation, any covariant derivative on E has the form

$$\phi \nabla_X \phi^{-1} u = X(u) + c_X \cdot u$$

for some $c_X \in C^\infty(U, \mathbb{C}^{r \times r})$ and any $u \in C^\infty(U, \mathbb{C}^r)$.

The notion of order gives a filtration of the algebra of differential operators in the usual way:

$$\mathcal{D}_H^k(E) = \{ P \colon \Gamma(E) \to \Gamma(E) \mid P \text{ is a differential operator of Heisenberg order } k\}$$

The principal symbol map is then the usual projection map

$$\sigma_H^k \colon \mathcal{D}_H^k(E) \to \mathcal{S}_H^k(E) := \mathcal{D}_H^k(E) / \mathcal{D}_H^{k-1}(E),$$

i.e. the mapping that "forgets" the parts of lower order, and one easily checks that the algebra structure (given by composition of operators) descends to one on

$$\mathcal{S}_H^*(E) = \bigoplus_k \mathcal{S}_H^k(E)$$

via $[P]_{\mathcal{S}} \cdot [Q]_{\mathcal{S}} = [P \circ Q]_{\mathcal{S}}$, which then automatically makes σ_H^* into an algebra homomorphism.

Remark. As already noted before, the commutator of two elements in H is not generally in H and thus, for $X_1, X_2 \in \Gamma(H)$,

$$\begin{aligned} \sigma_H^1(\nabla_{X_1}) \sigma_H^1(\nabla_{X_2}) &= \sigma_H^2(\nabla_{X_1} \nabla_{X_2}) \\ &= \sigma_H^2(\nabla_{X_2} \nabla_{X_1} + \nabla_{[X_1, X_2]} + R(X_1, X_2)) \\ &= \sigma_H^1(\nabla_{X_2}) \sigma_H^1(\nabla_{X_1}) + \sigma_H^2(\nabla_{[X_1, X_2]}). \end{aligned} \tag{4.10}$$

As $[X_1, X_2]$ generally has a transversal part, its second-order symbol is not zero, i.e. the composition of principal symbols is not commutative (R denotes the curvature endomorphism which vanishes going to principal symbols).

Convention. In what follows, when we speak about the order of an operator, we shall mean its Heisenberg order. If we need to refer to the order "in the usual sense", we will point that out.

In the remainder of the section, we will give a more concrete interpretation of the Heisenberg symbol. We begin by making the principal symbols local and then reinterpret the local symbols. In what way can we consider a symbol locally? To start with, we can say that two differential operators of order k, P and Q agree at a point a, if $P(e)(a) = Q(e)(a)$ for any $e \in \Gamma(E)$. Now, once we've gone to principal symbols, differences of lower order cannot be detected any more, so the best we can do is say whether two classes of operators agree in their highest order parts, i.e. their difference is a class of operators of order $\leq k$ that vanish at the point a. We formalise this in the following way: Define

$$\mathcal{S}^k_{H,a}(E) = \operatorname{span}\left\{ [FP]_{\mathcal{S}} \;\middle|\; F \in \Gamma(End(E))\colon F(a) = 0,\, P \in \mathcal{D}^k_H(E) \right\}.$$

Using that locally, the action of an endomorphism section F is given by matrix multiplication, one deduces that $PF = FP + LOT$, i.e. differential operators commute with sections of endomorphisms up to terms of lower order and thus, sections of endomorphisms commute with elements in $\mathcal{S}^*_H(E)$, making $\mathcal{S}^*_{H,a}(E)$ into an ideal in $\mathcal{S}^*_H(E)$. Thus, we can factor out the symbols vanishing at a and obtain the quotient

$$\mathcal{U}^k_a(E) = \mathcal{S}^k_H(E) / \mathcal{S}^k_{H,a}(E).$$

We will denote the projection of (the symbol class of) an operator P to $\mathcal{U}^k_a(E)$ by $[P]^k_a$. The algebra structure on $\mathcal{S}_H(E)$ induces one on $\mathcal{U}_a(E) = \bigoplus_k \mathcal{U}^k_a(E)$ via

$$[\sigma_H(P)]^k_a \cdot [\sigma_H(Q)]^l_a = [\sigma_H(P \circ Q)]^{k+l}_a.$$

To see that this is well-defined, assume that $P, \hat{P}$ are differential operators of order k and $Q, \hat{Q}$ of order l such that P and $\hat{P}$ as well as Q and $\hat{Q}$ define the same classes in $\mathcal{U}^*_a(E)$. Then, there exist endomorphism sections F_j, G_j satisfying $F_j(a) = G_j(a) = 0$ and differential operators A_j of order k and B_j of order l such that

$$\hat{P} = P + \sum_j F_j A_j + LOT \quad \text{and} \quad \hat{Q} = Q + \sum_j G_j B_j + LOT$$

Then (cf [vE05, p. 37],

$$\begin{aligned}
\hat{P}\hat{Q} &= PQ + \sum_j PG_j B_j + \sum_j F_j A_j Q + \sum_{j,k} F_j A_j G_k B_k + LOT \\
&= PQ + \sum_j (G_j P B_j + F_j A_j Q) + \sum_{j,k} F_j G_k A_j B_k + LOT.
\end{aligned}$$

As all the elements under the sums belong to $\mathcal{S}^{k+l}_{H,a}(E)$, the equivalence classes of $\hat{P}\hat{Q}$ and PQ agree in $\mathcal{U}^{k+l}_a(E)$.

Then, there is a one-to-one correspondence between elements of $\mathcal{S}^k_H(E)$ and sections of $\mathcal{U}^k(E) = \coprod \mathcal{U}^k_a(E)$, given by identifying $\sigma^d_H(P)$ with the section that at each points a evaluates to the image of the projection of $\sigma^k_H(P)$ to $\mathcal{U}^k_a(E)$. This identification induces the structure of a graded algebra on $\mathcal{U}(E) = \bigoplus_k \mathcal{U}^k(E)$.

We will now describe the pointwise symbol algebra $\mathcal{U}_a(E)$ through the enveloping algebra of $\mathfrak{t}_H\mathfrak{m}_a$.

Lemma 4.3.1 ([vE05, Prop. 38])**.** *There is an isomorphism of graded algebras $\chi\colon \mathcal{U}(\mathfrak{t}_H\mathfrak{m}_a)\otimes End(E_a) \to \mathcal{U}_a(E)$, where $\mathcal{U}(\mathfrak{t}_H\mathfrak{m}_a)$ is the universal enveloping algebra of the tangent Lie algebra $\mathfrak{t}_H\mathfrak{m}_a$ with a grading induced by that of $\mathfrak{t}_H\mathfrak{m}_a$. In a basis $Z_1, ..., Z_d$ of H_a, Z_0 transversal, χ is given by*

$$\chi\left(\sum_\gamma Z^\gamma \otimes C_\gamma\right) = \sum_\gamma [c_\gamma \cdot (\nabla_X)^\gamma]^{\langle\gamma\rangle}_a, \tag{4.11}$$

where $X_0, X_1, ..., X_d$ is a local H-frame such that $X_j(a) = Z_j$ for $j = 1, \ldots, d$, $[X_0(a)]_{H_a} = Z_0$ and $c_\gamma \in \Gamma(End(E))$ such that $c_\gamma(a) = C_\gamma$. The multiindex notation for ∇ is the same as for vector fields, i.e. by $(\nabla_X)^\gamma$, we mean $(\nabla_{X_0})^{\gamma_0}\cdots(\nabla_{X_d})^{\gamma_d}$ and $(\nabla_{X_j})^{\gamma_j}$ means derive γ_j times in the direction of X_j.

Proof. We begin by proving that the mapping is well-defined on $\mathfrak{t}_H\mathfrak{m}_a$. The difference between different extensions of Z_j and C_γ vanishes when considering classes in $\mathcal{U}_a(E)$ because they agree at a of the appropriate order. Next, we see that the mapping obviously preserves the grading and is linear. Concerning the multiplication structure, we note that the mapping is multiplicative by definition for products Z_jZ_k where $j < k$. For $j > k$ and $j, k \in \{1, \ldots, d\}$, we have

$$\begin{aligned}\chi(Z_jZ_k) &= \chi(Z_kZ_j + [Z_j, Z_k])\\ &= \chi(Z_kZ_j + L_{jk}Z_0)\\ &= \left[\nabla_{X_k}\nabla_{X_j} + L_{jk}\nabla_{X_0}\right]^2_a.\end{aligned}$$

As $([X_j, X_k] - L_{jk}X_0)(a) \in H_a$, $[L_{jk}\nabla_{X_0}]^2_a = [\nabla_{[X_j,X_k]_{\mathfrak{X}(M)}}]^2_a$ and thus, we have

$$\begin{aligned}\chi(Z_jZ_k) &= \left[\nabla_{X_k}\nabla_{X_j} + \nabla_{[X_j,X_k]_{\mathfrak{X}(M)}}\right]^2_a\\ &= \left[\nabla_{X_j}\nabla_{X_k} - R(X_j, X_k)\right]^2_a\\ &= \chi(Z_j)\chi(Z_k).\end{aligned}$$

If $k = 0$, we have $[Z_j, Z_0] = 0$ and thus

$$\begin{aligned}\chi(Z_j Z_0) = \chi(Z_0 Z_j) &= \big[\nabla_{X_j}\nabla_{X_0}\big]_a^3 \\ &= \Big[\nabla_{X_j}\nabla_{X_0} - R(X_j, X_0) - \nabla_{[X_j,X_0]_{\mathfrak{X}(M)}}\Big]_a^3,\end{aligned}$$

where the last equality holds because all additional terms are of order less than three. The right-hand side is then equal to $[\nabla_{X_0}\nabla_{X_j}]_a^3$ and thus to $\chi(Z_j)\chi(Z_k)$. Hence, we obtain an algebra morphism. Finally, choosing a basis (C_{kl}) of endomorphism of E_a and extensions (c_{kl}), the enveloping algebra is generated by $(Z_j \otimes C_{kl})$ as an algebra and $\mathcal{U}_a(E)$ by $c_{kl} \cdot \nabla_{X_j}$, and thus, χ is bijective. □

We can now consider the Heisenberg symbol $\sigma_H^k(P)$ as a section of the bundle of enveloping algebras (tensored with bundle endomorphisms). These sections can be interpreted in two ways: First, they can be seen as polynomial functions in $\mathfrak{t}_H\mathfrak{m}_a^*$, an interpretation close to the interpretation of the usual symbols of differential operators as sections of the bundle of symmetric tensors $S(TM)$. Second, elements of the Lie algebra $\mathfrak{t}_H\mathfrak{m}_a$ can be interpreted as left-invariant vector fields on $T_H M_a$ and thus, we can consider the elements of the universal enveloping algebra as left-invariant differential operators on the trivial bundle $T_H M_a \times E_a$. These operators, which we will call "model operators" play an important role in determining the analytic properties of a differential operator in the Heisenberg calculus.

We begin with the interpretation as invariant differential operators.

Proposition 4.3.2. *The Heisenberg principal symbol σ_H induces a morphism of graded algebras $\sigma_H\colon \mathcal{D}_H(E) \to \Gamma(\mathcal{U}(\mathfrak{t}_H\mathfrak{m}) \otimes End(E))$. If P is a differential operator of order k locally around a given by (4.9), its principal symbol $\sigma_H^k(D)(a)$ induces a left-invariant differential operator P^a on $T_H M_a \times E_a$ that is given by*

$$P^a = \sum_{\langle\gamma\rangle = k} (X^a)^\gamma \otimes b_\gamma(a),$$

where X_j^a are the left-invariant vector fields on $T_H M_a$ defined by X_j.

Definition. The operator P^a is called the *model operator* of P at the point $a \in M$.

We now turn to the interpretation as polynomial functions. The usual principal symbol is an element of $S^k(TM) \otimes \mathrm{End}(E)$, where S^k is the k-th order part of the symmetric algebra. This algebra may be interpreted as formal homogeneous polynomials in TM, or, equivalently, as homogeneous polynomial functions in T^*M. In the same way, $\mathcal{U}(\mathfrak{t}_H\mathfrak{m})$ are formal polynomials in $\mathfrak{t}_H\mathfrak{m}$, where the product structure and notion of degree have been adapted to the graded case. In other words, we can consider the Heisenberg principal symbol of a differential operator of (Heisenberg) order k as a section $\sigma_H(D) \in \Gamma(\mathfrak{t}_H\mathfrak{m}^*, \mathrm{End}(\pi^*E))$ (where π is the projection $\pi\colon \mathfrak{t}_H\mathfrak{m}^* \to M$) that is polynomial of graded degree k in $\mathfrak{t}_H\mathfrak{m}^*$.

Note that in the above definition of the isomorphism $\mathcal{U}(\mathfrak{t}_H\mathfrak{m}_a)\otimes End(E_a) \simeq \mathcal{U}_a(E)$ we left out the usual factor i because we wanted to directly interpret the result as a differential operator again (going to a symbol and back to a differential operator would cancel out the factor i). If we want to stay closer to the usual definition of principal symbols (and the definition for pseudodifferential operators in the following section), we will need to map Z_j to $-i\nabla_{X_j}$ in (4.11) instead. In this sense, under the conditions of Proposition 4.3.2, the principal symbol induces an injective map

$$\sigma_H^*\colon \mathcal{S}_H^*(E) \to \Gamma(\mathfrak{t}_H\mathfrak{m}^*, \mathrm{End}(\pi^*E))$$

given as follows: For a differential operator D of (Heisenberg) order k, given locally by (4.9), $\sigma_H^k(D) \in \Gamma(\mathfrak{t}_H\mathfrak{m}^*, \mathrm{End}(\pi^*E))$ is locally defined by interpreting the formal polynomial

$$\sigma_H^k(D)(a) = \sum_{\langle\gamma\rangle = k} (-iX^a)^\gamma \otimes b_\gamma(a) \tag{4.12}$$

as a function on $\mathfrak{t}_H\mathfrak{m}^*$ with values in $End(E)$.

This mapping is injective, but not an isomorphism. In fact, we obtain the subspace of $\Gamma(\mathfrak{t}_H\mathfrak{m}^*, \mathrm{End}(\pi^*E))$ given by polynomial functions. If we extend this to homogeneous functions (in an appropriate sense), we obtain the principal symbols of the *pseudodifferential* operators that we will introduce in the following section.

As for elliptic operators, the symbol (or the model operator) can be used to determine analytic properties of the operator. This will be discussed in section 4.5. Before, we will extend the Heisenberg calculus to pseudodifferential operators, which will provide parametrices for certain Heisenberg differential operators. While these are important for the theory behind the hypoellipticity criterion, the criterion may be understood without them and the casual reader may skip directly ahead to section 4.5. When coming across principal symbols in that section, bear in mind the necessary change of the identification of the abstract symbol class with $\mathcal{U}(\mathfrak{t}_H\mathfrak{m}_a) \otimes End(E_a)$ mentioned above.

4.4 Heisenberg pseudodifferential operators

The theory of Heisenberg pseudodifferential operators will be developed in three steps:

- Invariant pseudodifferential operators on nilpotent groups (recall that this is the tangent structure for a Heisenberg manifold). These operators will be the model operators for pseudodifferential operators on M.

- Heisenberg pseudodifferential operators on open subsets of $\mathbb{R}^n$ (the local model for pseudodifferential operators on manifolds)
- Heisenberg pseudodifferential operators on manifolds.

Before we begin the discussion in detail, we give a (very) brief motivation. Consider a differential operator of (usual) order k on functions on $\mathbb{R}^n$, given by

$$(Pf)(x) = \sum_{|\gamma| \leq k} b_\gamma(x) D^\gamma f(x), \quad \text{where } D^\gamma = (-i)^{|\gamma|} \frac{\partial^{|\gamma|}}{\partial x^\gamma}. \tag{4.13}$$

Then, using the properties of Fourier transform, this operator can be written as

$$Pf(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi) \hat{f}(\xi) d\xi \tag{4.14}$$

for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^n)$, where $\hat{f}$ is the Fourier transform of f, i.e.

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x) dx,$$

and p, the *full symbol* of P, is a polynomial in ξ given by

$$p(x,\xi) = \sum_{|\gamma| \leq k} b_\gamma(x) \xi^\gamma,$$

i.e. formally $P = p(x, D)$. Whereas we went from operator to symbol here, for pseudodifferential operators we go in the opposite direction. We replace the polynomial p by a more general function, like a (possibly infinite) sum of homogeneous functions (these would yield the so-called classical pseudodifferential operators) or a function with similar growth behaviour as $\xi \to \infty$, and then define an operator via (4.14). The differential operators of order k are always contained in the pseudodifferential operators of the same order. There are many different classes of pseudodifferential operators, cf [Tay81] or [Shu01].

In the Heisenberg calculus, theses classes of functions (or symbols) are replaced with ones that are more adapted to the Heisenberg calculus, in particular, a differential operator of Heisenberg order k should be a Heisenberg pseudodifferential operator of order k. Locally, there is not much of a difference between bundle-valued and scalar operators as the former are simply matrices of the latter. Therefore, in the next two sections, we will focus on scalar operators and return to bundle-valued operators only in section 4.4.3, when we consider operators on manifolds. Our main sources are [BG88] (and occasionally [CGGP92]) for the local theory and [Pon08] for the global calculus on manifolds.

4.4.1 Invariant pseudodifferential operators on the tangent group

The first class of symbols we will consider are those of left-invariant pseudodifferential operators on a nilpotent group. Just like the differential operators we have seen above, every pseudodifferential operator on a Heisenberg manifold is locally "modelled on" such an invariant operator.

We begin by introducing the appropriate classes of symbols that define the homogeneous left-invariant pseudodifferential operators. We then discuss their representation by kernels instead of symbols and end this section with a discussion of their product structure.

As said before, the class of symbols of pseudodifferential operators should extend the class of symbols of the differential operators. We consider $\mathfrak{t}_H\mathfrak{m}_a$ ($a \in M$) equipped with a basis of left-invariant vector fields $X_0^a, X_1^a, \ldots, X_d^a$ with $X_1^a, \ldots, X_d^a$ spanning H_a. We identify $\mathfrak{t}_H\mathfrak{m}_a$ with $\mathbb{R}^{d+1}$ via the linear isomorphism (4.2) induced by the basis. A homogeneous (in the appropriate sense, which we will make precise later) left-invariant differential operator of degree k is a homogeneous polynomial in left-invariant vector fields, so with $X_0^a, X_1^a, \ldots, X_d^a$ as above

$$(Pf)(x) = p(-iX^a)f(x) = \sum_{\langle\gamma\rangle=k} b_\gamma \cdot ((-iX^a)^\gamma f)(x) \tag{4.15}$$

for $f \in C^\infty(T_H M_a)$, i.e. p is a polynomial that is homogeneous with respect to the multiplication $\lambda.\xi = (\lambda^2\xi_0, \lambda\xi_1, ..., \lambda\xi_d)$. Note that the b_γ are constants. We let σ_j^a denote the usual symbol of $-iX_j^a$ in the standard coordinates, i.e. we have (cf (4.5))

$$\sigma_0^a(x,\xi) = \xi_0 \quad \text{and} \quad \sigma_j^a(x,\xi) = \xi_j - \frac{1}{2}\sum_{k=1}^{d} L_{jk}(x)x_k\xi_0. \tag{4.16}$$

The form (4.14) is then obtained if we replace p by $p(\sigma^a(x,\xi))$, where $\sigma^a = (\sigma_0^a, \ldots, \sigma_d^a)$. This leads us to define

Definition.

1. The scalar multiplication structure on $\mathbb{R}^{d+1}$ given by
$$\delta_\lambda(\xi) = \lambda.\xi = (\lambda^2\xi_0, \lambda\xi_1, ..., \lambda\xi_d)$$
is called the *Heisenberg dilation.*
2. The norm $\|\xi\|_H = \left(\xi_0^2 + \left(\xi_1^2 + ... + \xi_d^2\right)^2\right)^{1/4}$ is called *Heisenberg norm.*
3. For $k \in \mathbb{Z}$, we denote by $S_k(\mathfrak{t}_H\mathfrak{m}_a)$ the class of functions $p \in C^\infty(\mathbb{R}^{d+1} \setminus \{0\})$ which are Heisenberg-homogeneous of degree k, i.e. $p(x, \lambda.\xi) = \lambda^k p(x,\xi)$ for any $\lambda > 0$. If no confusion about the manifold or point is possible, we will simply write S_k instead of $S_k(\mathfrak{t}_H\mathfrak{m}_a)$.

As the functions $p \in S_k$ have a singularity at 0 for $k < 0$, we will need to choose an appropriate test space to ensure that the associated pseudodifferential operators are well-defined. We define the following subspace of the space of Schwartz functions $\mathcal{S}(\mathbb{R}^{d+1})$ as introduced in [Pon08, section 3.1.1]:

$$\mathcal{S}_0 = \left\{ f \in \mathcal{S}(\mathbb{R}^{d+1}) \,\middle|\, P\hat{f}(0) = 0 \text{ for any differential operator } P \right\}.$$

We equip $\mathcal{S}_0$ with the topology induced from $\mathcal{S}(\mathbb{R}^{d+1})$. Using the Taylor expansion of a function, we see that $\hat{f}(\xi) = O(\|\xi\|^N)$ for $\xi \to 0$ and any $N \in \mathbb{N}$. This "smoothes away" the possible singularities if $p \in S^k$. Furthermore, differentiating under the integral sign in

$$\frac{\partial^{|\gamma|}}{\partial \xi^\gamma}\hat{f}(0) = \frac{\partial^{|\gamma|}}{\partial \xi^\gamma}\left(\int e^{i\langle x,\xi\rangle} f(x)dx\right)|_{\xi=0},$$

we obtain that

$$\mathcal{S}_0 = \left\{ f \in \mathcal{S}(\mathbb{R}^{d+1}) \,\middle|\, \int x^\gamma f(x)dx = 0 \text{ for all monomials } x^\gamma \right\},$$

the definition of $\mathcal{S}_0$ used in [CGGP92]. Here and in what follows, we omit the domain of integration if it is the whole euclidean space.

Lemma 4.4.1 ([Pon08, Prop 3.1.3 and comments thereafter])**.** *Let $p \in S_k$. Then, for any $f \in \mathcal{S}_0$, the integral*

$$Pf(x) := p(-iX^a)f(x) := (2\pi)^{-d-1}\int e^{i\langle x,\xi\rangle} p(\sigma^a(x,\xi))\hat{f}(\xi)d\xi \tag{4.17}$$

is well-defined and we obtain a continuous endomorphism $P\colon \mathcal{S}_0 \to \mathcal{S}_0$.

Definition. The operator $P = p(-iX^a)\colon \mathcal{S}_0 \to \mathcal{S}_0$ defined via (4.17) with $p \in S_k$ is called a *homogeneous left-invariant pseudodifferential operator of order* k and p is called its symbol. The class of homogeneous left-invariant pseudodifferential operators of order k is denoted by $\Psi_k(T_H M_a)$.

Remark (Differential operators)**.** We call a left-invariant differential operator P homogeneous of degree k on $T_H M_a$ if

$$P^{(\lambda)} := \delta_\lambda^{-1} \circ P \circ \delta_\lambda = \lambda^k P.$$

One may easily see that an invariant differential operator is homogeneous of degree k if and only if it is of the form (4.15). By the discussion following (4.15), we see that the class of homogeneous left-invariant differential operators is contained in $\Psi_k(T_H M_a)$ as we wanted.

We will now also consider left-invariant pseudodifferential operators that are not homogeneous, but have an asymptotic expansion in homogeneous symbols. We aren't interested in these operators as such (although this will be the type of operator we will consider in the non-invariant case), but we will need them for the discussion of products.

Definition. The class S^k consists of smooth functions $p \in C^\infty(\mathbb{R}^{d+1})$ with an asymptotic expansion

$$p \sim \sum_{j\geq 0} p_{k-j}, \qquad p_l \in S_l \quad (l = k, k-1, \dots)$$

in the sense that for any multi-index γ and any $N \in \mathbb{N}$ there exists a constant $C_{\gamma N}$ such that

$$\left| D_\xi^\gamma \left(p - \sum_{j<N} p_{k-j} \right) (\xi) \right| \leq C_{\gamma N} \|\xi\|_H^{k-N-\langle\gamma\rangle}. \tag{4.18}$$

Any function $p \in S^k$ defines a pseudodifferential operator $P = p(-iX^a) \colon \mathcal{S}(\mathbb{R}^{d+1}) \to \mathcal{S}(\mathbb{R}^{d+1})$ via (4.17). We denote the class of such operators $\Psi^k_{inv}(T_H M_a)$.

Lemma 4.4.2. *The asymptotic expansion of any $p \in S^k$ is unique.*

Proof. Assume we have two asymptotic expansions $p \sim \sum p_{k-j}$, $p \sim \sum q_{k-j}$. We begin by considering the first summand. Using the definition of an asymptotic expansion, we have

$$|(p - p_k)(\xi)| \leq C_p \|\xi\|_H^{k-1} \quad \text{and} \quad |(p - q_k)(\xi)| \leq C_q \|\xi\|_H^{k-1}$$

and thus, using homogeneity of the functions,

$$\begin{aligned} |\lambda|^k |p_k(\xi) - q_k(\xi)| =& |p_k(\lambda\xi) - q_k(\lambda\xi)| \\ \leq& |(p - p_k)(\lambda\xi)| + |(p - q_k)(\lambda\xi)| \\ \leq& (C_p + C_q) \|\lambda\xi\|_H^{k-1} \\ =& |\lambda|^{k-1} (C_p + C_q) \|\xi\|_H^{k-1} \end{aligned}$$

Sending $\lambda \to \infty$ yields $p_k - q_k = 0$. By induction, all summands can be seen to be equal. □

Remark.
- A function $f \in S^k$ satisfies the estimate (cf [BG88, Remark 12.7])

$$|D^\gamma f(\xi)| \leq C_\gamma (1 + \|\xi\|_H)^{k-\langle\gamma\rangle}. \tag{4.19}$$

- Obviously, $S^k \subset S^{k+1}$. If $f \sim \sum f_{k-j}$, then $f \sim \sum f_{k+1-j}$ with $f_{k+1} = 0$.

- The elements of $S^{-\infty} = \bigcap_{k\in\mathbb{Z}} S^k$ are exactly those functions $f \in C^\infty(\mathbb{R}^{d+1})$ that have an expansion $f \sim 0$: As $f \in S^k$ for any k, by the previous remark, it has an expansion starting with the term f_{k-1} for any k. Conversely, $0 \in S_k$ for all k and thus $f \in S^{-\infty}$ if $f \sim 0$.

Lemma 4.4.3 ([BG88, Prop 12.9])**.** *Let $p \in C^\infty(\mathbb{R}^{d+1})$ satisfy* (4.19)*. Then, p defines an operator $p(-iX^a)\colon \mathcal{S}(\mathbb{R}^{d+1}) \to \mathcal{S}(\mathbb{R}^{d+1})$ via* (4.17)*. The operator determines its symbol, i.e. $p(-iX^a) = 0$ if and only if $p = 0$.*

Kernels of homogeneous left-invariant operators

Any pseudodifferential operator of the form (4.17) may also be represented by its kernel, i.e. formally written as

$$Pf(x) = \int_{\mathbb{R}^{d+1}} K(x, x-y) f(y) dy, \tag{4.20}$$

where, formally, the kernel K is obtained as the inverse Fourier transform of the symbol p in the second variable. Now, the Fourier transform is not defined for the functions in our symbol class S_k. We may, however, obtain a Fourier transform if we extend them to distributions on $\mathbb{R}^{d+1}$ as follows.

While the kernels are important as a tool to prove many results on left-invariant pseudodifferential operators, most of the theory can be understood without them and this section may be skipped by the casual reader.

Definition.

1. A distribution $\varphi \in \mathcal{S}'(\mathbb{R}^{d+1})$ is called *homogeneous of degree k* if $\varphi_\lambda = \lambda^k d$, where
$$\langle \varphi_\lambda, f\rangle = \lambda^{-(d+2)}\langle \varphi, f(\lambda^{-1}.\cdot)\rangle.$$
Note that with this definition, a smooth distribution is homogeneous if and only if it is homogeneous as a function.

2. A distribution $\varphi \in \mathcal{S}'(\mathbb{R}^{d+1})$ is called *regular* if it is smooth outside the origin. We note the space of regular tempered distributions $\mathcal{S}'_\infty(\mathbb{R}^{d+1})$.

3. We denote by $\mathcal{G}_k$ the space of elements $K \in \mathcal{S}'_\infty(\mathbb{R}^{d+1})$ with the following property: There exist constants c_γ such that
$$K_\lambda = \lambda^k K + \sum_{\langle\gamma\rangle = -k-2m-2} c_\gamma(\lambda^k \ln\lambda)\delta^{(\gamma)},$$
where $\delta^{(\gamma)}$ are the derivatives of the Dirac distribution, i.e. $\delta^{(\gamma)} f = \frac{\partial^{|\gamma|} f}{\partial x^\gamma}(0)$.

As it turns out, this class contains exactly the extensions of symbols of type S_k:

Proposition 4.4.4 ([BG88, Prop 15.6]). *Let $K \in \mathcal{G}_k$. Then, the restriction of K to $\mathbb{R}^{d+1} \setminus \{0\}$ is in S_k. Conversely, for any $p \in S_k$, there exists a $K \in \mathcal{G}_k$ that agrees with p on $\mathbb{R}^{d+1} \setminus \{0\}$. Such a distribution may be defined via*

$$\langle K, u \rangle = \int p(\xi) \left(u(\xi) - \sum_{0 \le \langle \gamma \rangle \le -k-d-2} \frac{u^{(\gamma)}(0)}{\gamma!} \xi \gamma^{\phi}(\xi) \right) d\xi,$$

where $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ is some function that is equal to one near the origin.

Note that in [BG88], $\mathcal{G}_k$ is defined as a subspace of $\mathcal{D}'(\mathbb{R}^{2m+1})$. Its properties make any distribution in it into a tempered one however, as remarked by Beals and Greiner after definition (15.18).

As the inverse Fourier transform of tempered distributions is well-defined, we may now determine the class of distributional kernels of homogeneous left-invariant pseudodifferential operators.

Definition. The class $\mathcal{K}_k$ consists of tempered distributions $K \in \mathcal{S}'_\infty(\mathbb{R}^{d+1})$ satisfying

$$K_\lambda = \lambda^k K + (\lambda^k \ln \lambda) \cdot c, \tag{4.21}$$

where c is some polynomial that is homogeneous of degree k with respect to the Heisenberg dilations.

Proposition 4.4.5 ([BG88, Prop 15.21, 15.24]).

1. *Any $K \in \mathcal{K}_k$ is of the form*

   $$K(x) = p(x) + b(x) \ln \|x\|_H,$$

   where $p \in S_k$ and b is a polynomial homogeneous of degree k in the Heisenberg sense.

2. *The inverse Fourier transform is a bijection from $\mathcal{G}_k$ to K_{-k-d-2}.*

Remark. One may topologise the distribution spaces considered above and can show that the bijection from $\mathcal{G}_k$ to K_{-k-d-2} is actually a homeomorphism. The interested reader may find the details in [BG88, pp. 131-132]

From the above results, we can now deduce the kernel of a homogeneous left-invariant pseudodifferential operator. What remains to do is to see what becomes of the term $\sigma^a(x, \xi)$. Recalling (4.16), we may write σ^a as a matrix product $\sigma^a(x, \xi) = B_x \xi$, where $B_x = (b_{jl}(x))$ and $X_j^a(x) = \sum_{l=0}^d b_{jl}(x) \frac{\partial}{\partial x_l}$. Then, the inverse Fourier transform of $p(\sigma^a(x, \xi))$ in ξ (interpreted as a distribution) yields $det(B_x^{-1}) K(x, (B_x^T)^{-1}(x - y))$, where $K(x, y)$ is the inverse Fourier transform of p (cf [Pon08, (3.1.27)]). Using this and the above proposition, we obtain the following result (cf also [BG88, Prop (15.39) and (15.49)]).

Proposition 4.4.6. *A homogeneous left-invariant pseudodifferential operator $P \in \Psi_k(T_H M_a)$ has a distributional kernel of the form*

$$K(x,y) = \det\left((B_x)^{-1}\right) L((B_x^T)^{-1}(x-y)),$$

where $L \in \mathcal{K}_{-k-d-2}$ and $B_x = (b_{jl}(x))$ is the matrix of coefficients of X^a, i.e. $X_j^a(x) = \sum_{l=0}^d b_{jl}(x) \frac{\partial}{\partial x_k}$.

Conversely, for a kernel K of the above form, the operator $\mathfrak{K} f = K \star f$ (where $\star$ denotes convolution) is a homogeneous left-invariant pseudodifferential operator of order k.

Remark. From this reformulation, we see that our homogeneous left-invariant pseudodifferential operator of order k are exactly the operators $\mathcal{O}_0(K)$ for $K \in \mathbb{K}^k$ of [CGGP92].

We move on to the discussion of kernels of operators $p(-iX^a)$ for $p \in S^k$. Just like the symbols have an asymptotic expansion in symbols $p_l \in S_l$, the distributional kernel will have an expansion in distributions in $\mathcal{K}_k$.

Definition. The class $\mathcal{K}^k$ consists of distributions $K \in \mathcal{D}'(\mathbb{R}^{d+1})$ with an asymptotic expansion

$$K \sim \sum_{j=0}^{\infty} K_{k+j}, \quad K_{k+j} \in \mathcal{K}_{k+j},$$

in the sense that for every $N > 0$ there exists some $J > 0$ such that

$$K - \sum_{j=0}^{J} K_{k+j} \in C^N(\mathbb{R}^{d+1}).$$

We then have the following characterisation of invariant pseudodifferential operators via their kernels:

Proposition 4.4.7 ([BG88, Prop 15.39, 15.49])**.** *A left-invariant pseudodifferential operator $P \in \Psi_{inv}^k(T_H M_a)$ has a distributional kernel of the form*

$$K(x,y) = \det\left((B_x)^{-1}\right) L((B_x^T)^{-1}(x-y)),$$

where $L \in \mathcal{K}^{-k-d-2}$ and $B_x = (b_{jl}(x))_{jl}$ is the matrix of coefficients of X^a, i.e. $X_j^a(x) = \sum_{l=0}^d b_{jl}(x) \frac{\partial}{\partial x_k}$.

Conversely, for a kernel K of the above form, the operator $\mathfrak{K} f = K \star f$ (where $\star$ denotes convolution) is a left-invariant pseudodifferential operator of order k.

Proof. The result in the book of Beals and Greiner is stated for non-invariant operators. However, the difference is only an additional dependence of the symbols and kernels on the point x. As we let the Fourier transform act in the second (the ξ-)variable to makes symbols into kernels and vice versa, analogous results hold for the invariant operators. □

Products of homogeneous left-invariant pseudodifferential operators

For a useful symbol calculus, we want the product (i.e. the composition) of two ΨDOs to be a ΨDO again and their (principal) symbol to be given by some sort of product of the symbols. This is indeed the case for homogeneous left-invariant ΨDOs. Unfortunately, defining the product of symbols is rather involved. A formal calculation shows that a symbol product # satisfying this must be of the following form

$$(p\#q)(\xi) = (2\pi)^{-d-1} \iint e^{-i\langle z,\eta\rangle} p(\xi+\eta) q(\sigma^a(z,\xi)) dz d\eta \tag{4.22}$$

Obviously, for $p, q \in S_*$, this integral is anything but convergent. The rest of this section will be devoted to a discussion of the existence of such a product. The reader who is willing to believe us that such a product exists may skip it.

Theorem 4.4.8. *There exists a unique product*

$$\#_a \colon S_{k_1}(\mathfrak{t}_H\mathfrak{m}_a) \times S_{k_2}(\mathfrak{t}_H\mathfrak{m}_a) \to S_{k_1+k_2}(\mathfrak{t}_H\mathfrak{m}_a)$$

such that

$$p_1(-iX^a) \circ p_2(-iX^a) = (p_1\#_a p_2)(-iX^a). \tag{4.23}$$

Proof. By the results of the section above, each of the operators $p_j(-iX^a)$ is given by a kernel $K_j \in \mathcal{K}_{-k_j-d-2}$. By Proposition 2.3 of [CGGP92], there exists a product $K_1 \underline{\star} K_2 \in \mathcal{K}_{-(k_1+k_2)-d-2}$ that is the kernel of the product of the operators. The inverse Fourier transform of $K_1 \underline{\star} K_2$ is the the symbol of the product of operators, i.e. $p_1\#_a p_2 = \mathcal{F}^{-1}(K_1 \underline{\star} K_2)|_{\mathbb{R}^{d+1}\setminus\{0\}}$.

While there is a choice involved in the definition $K_1 \underline{\star} K_2$, any other kernel of $p_1(-iX^a) \circ p_2(-iX^a)$ differs by a homogeneous polynomial. As the Fourier transform (on the space of tempered distributions) maps polynomials to sums of Dirac distributions, this difference does not appear in the symbols (which are restrictions to $\mathbb{R}^{d+1} \setminus \{0\}$). Thus, $\#_a$ is uniquely determined by (4.23). □

In [BG88], there is also a product structure for homogeneous symbols. However, it remains unclear whether that product is the symbol of the product of operators. As we will use results for this product in the future, we would like to prove that it coincides with the one defined via kernels above and thus, that the product of symbols is the symbol of the product (composition) of operators.[1]

Beals and Greiner begin by introducing a product for symbols of class S^k. For these symbols, a formal calculation shows that a symbol $\#_a^{BG}$ satisfying (4.23) must be of the form (4.22). The integral in that formula is not a priori convergent. To

[1] The author would like to thank Raphaël Ponge for very helpful communication on the subject.

circumvent this problem, we can introduce a smooth cutoff function $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ with $\phi \equiv 1$ near zero, set

$$\phi_\varepsilon(z,\xi,\eta) = \phi\left(\varepsilon\left(1 + \|z\|^2 + \|\eta\|_H^4 + \|\sigma^a(z,\xi)\|_H^4\right)\right)$$

and consider

$$(p\#_a^{BG}q)(\xi) = \lim_{\varepsilon\to 0^+}(2\pi)^{-d-1}\iint \phi_\varepsilon(z,\xi,\xi+\eta)e^{-i\langle z,\eta\rangle}p(\xi+\eta)q(\sigma^a(z,\xi))dzd\eta.$$

For details, see the proof of theorem 12.14 in [BG88].

Proposition 4.4.9 ([BG88, section 12])**.** *For $p \in S^k$ and $q \in S^l$, the product* (4.22) *is well-defined, $p\#_a^{BG}q \in S^{k+l}$ and satisfies* (4.23)*.*

We now want to use this product to define one for homogeneous symbols in S_k. To begin with, we can define a function $f \in S^k$ with asymptotic expansion $f \sim p$ for any $p \in S_k(\mathfrak{t}_H\mathfrak{m}_a)$ as follows: Fix a smooth function ϕ with $\phi(\xi) = 0$ for $\|\xi\| < \frac{1}{2}$ and $\phi(\xi) = 1$ for $\|\xi\| > 1$ and set $f(\xi) = p(\xi)\phi(\xi)$. Then, $f - p = p(\phi - 1)$ and for large ξ, the difference vanishes. For $\|\xi\| < \frac{1}{2}$ on the other hand, $(f-p)(\xi) = p(\xi)$ and (4.18) for $\gamma = 0$ follows from homogeneity of p. For $\langle\gamma\rangle > 0$, we have $D_\xi^\gamma(f-p) = (D_\xi^\gamma p)(\phi-1) + pD_\xi^\gamma\phi$ and make use of the facts that $\phi - 1$ is zero for large ξ, $D_\xi^\gamma\phi$ is zero both near zero and infinity and the derivatives of p are homogeneous of order $k - \langle\gamma\rangle$.

If we choose another $\hat{\phi}$, we obtain another function $\hat{f} \in S^k$, $\hat{f} \sim p$. Then the function s defined by $s(\xi) := f - \hat{f}(\xi) = (\phi - \hat{\phi})(\xi)$ is zero both near 0 and infinity and thus, $s \sim 0$ and therefore $s \in S^{-\infty} = \bigcap_{k\in\mathbb{Z}} S^k$.

Now, for $p_j \in S_{k_j}(\mathfrak{t}_H\mathfrak{m}_a)$ $(j = 1,2)$, we choose $f_j \in S^{k_j}$ such that $f_j \sim p_j$. One may then show that there is a unique $r \in S_{k+l}(\mathfrak{t}_H\mathfrak{m}_a)$ that provides the expansion of $(f_1\#_a^{BG}f_2)$, i.e. $(f_1\#_a^{BG}f_2) \sim r$ (compare [BG88, Prop 12.72, Thm 12.82]). Indeed, one may explicitly obtain r as $r(\xi) = \lim_{\lambda\to\infty}\lambda^{-k}(f_1\#_a^{BG}f_2)(\lambda.\xi)$. Thus, we define the product $p_1\#_a^{BG}p_2$ as follows:

$$(p_1\#_a^{BG}p_2) = \lim_{\lambda\to\infty}\lambda^{-k}(f_1\#_a^{BG}f_2)(\lambda.\xi). \tag{4.24}$$

Lemma 4.4.10. *The product* (4.24) *is independent of the choice of smooth functions f_j.*

Proof. Assume $\hat{f}_j \in S^{k_j}$ such that $\hat{f}_j \sim p_j$. Then $\hat{f}_j = f_j + s_j$ with $s_j \in S^{-\infty}$. The product $\#_a^{BG}$ is obviously linear and thus

$$\hat{f}_1\#_a^{BG}\hat{f}_2 = (f_1\#_a^{BG}f_2) + \underbrace{(f_1\#_a^{BG}s_2) + (s_1\#_a^{BG}f_2) + (s_1\#_a^{BG}s_2)}_{=s}.$$

Because $s_j \in S^k$ for all $k \in \mathbb{Z}$, we have that $f_j \#_a^{BG} s_j$ and $s_1 \#_a^{BG} s_2$ are in S^k for any $k \in \mathbb{Z}$ and therefore $s \in S^{-\infty}$. Then, $s \sim 0$ and $(\hat{f}_1 \#_a^{BG} \hat{f}_2) \sim r + 0$ and because the expansion is unique, r is the only possible expansion of $(\hat{f}_1 \#_a^{BG} \hat{f}_2)$ and setting $(p_1 \#_a^{BG} p_2) := r$ is well defined. □

We now want to show that the two products agree: $p_1 \#_a^{BG} p_2 = p_1 \#_a p_2$. Let $K_j \in \mathcal{K}_{k_j}$ be the distributional kernel of the operator $p_j(-iX^a)$ and let $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ and $\phi \equiv 1$ near zero. Then, $K_j - \phi K_j = (1-\phi)K_j$ is zero near 0 and smooth away from zero by the definition of $\mathcal{K}_{k_j}$. Thus, $\phi K_j \sim K_j \in \mathcal{K}^{-k_j-d-2}$, and by Proposition 4.4.7, the convolution operators $\mathfrak{K}_j$ defined by them are in $\Psi_{inv}^{k_j}(T_H M_a)$. We denote $f_j \in S^k$ their symbols. As the distributions ϕK_j are compactly supported, the convolution is associative and we obtain

$$\mathfrak{K}_1 \mathfrak{K}_2 u = \phi K_1 \star (\phi K_2 \star u) = (\phi K_1 \star \phi K_2) \star u.$$

By [CGGP92, Prop 3.3], we have that $(\phi K_1 \star \phi K_2) - (K_1 \underline{\star} K_2) \in C^\infty(\mathbb{R}^{d+1})$, i.e.

$$(\phi K_1 \star \phi K_2) \sim K_1 \underline{\star} K_2 \in \mathcal{K}^{-k_1-k_2-d-2}.$$

By Proposition 4.4.7, the operator $\mathfrak{K}_1 \mathfrak{K}_2$ has symbol

$$f_1 \#_a^{BG} f_2 \sim p_1 \#_a^{BG} p_2 \in S^{k_1+k_2}$$

and thus, the two products for symbols agree.

4.4.2 Pseudodifferential operators – local theory

Now let us move on to the Heisenberg calculus (rather, a local version of it), where we have the following situation:

Situation 4.4.11. *Let $U \subset \mathbb{R}^{d+1}$ be open, $H = \operatorname{span}\{X_1, ..., X_d\}$ be a codimension one distribution in TU and $X_0 \perp H$ everywhere nonzero. Let $X_j = \sum_k X_j^k(x) e_k$ and $\sigma_j(x,\xi) = \sum_k i X_j^k \xi_k$. Furthermore, we write $\mathbb{R}_0^{d+1}$ for $\mathbb{R}^{d+1} \setminus \{0\}$.*

The pseudodifferential operators here are very similar to the invariant ones considered above and the treatment will follow the same outline as before: Definition and basic properties, characterisation through kernels and finally products. The main difference is that the operators are not left-invariant anymore, so their symbols will depend on the point $x \in U$. Moreover, we are now mainly interested not in operators with homogeneous symbols but in operators with an asymptotic expansion in homogeneous symbols.

Definition. Assume situation 4.4.11.

1. The scalar multiplication structure given by
$$\delta_\lambda(\xi) = \lambda.\xi = (\lambda^2\xi_0, \lambda\xi_1, ..., \lambda\xi_d)$$
is called the *Heisenberg dilation*.

2. The norm $\|\xi\|_H = \left(\xi_0^2 + \left(\xi_1^2 + ... + \xi_d^2\right)^2\right)^{1/4}$ is called *Heisenberg norm*.

3. For $k \in \mathbb{Z}$, we denote by $S_k(U)$ the class of functions $p \in C^\infty(U \times \mathbb{R}_0^{d+1})$ which are Heisenberg-homogeneous in the second argument, i.e. $p(x, \lambda.\xi) = \lambda^k p(x, \xi)$ for any $\lambda > 0$. To avoid confusion with the invariant symbols, we will never omit the argument U in $S_k(U)$.

4. For $k \in \mathbb{Z}$, the class $S^k(U)$ consisting of functions $p \in C^\infty(U \times \mathbb{R}_0^{d+1})$ with an asymptotic expansion
$$p \sim \sum_{j=0}^{\infty} p_{k-j}, \qquad p_l \in S_l(U)$$
in the sense that for all multi-indices β, γ, all $N \in \mathbb{N}$ and all compact sets $K \subset U$ there is a constant $C = C(\beta, \gamma, N, K)$ such that for all $x \in K, \xi \in \mathbb{R}^{d+1}$,
$$\left| D_x^\gamma D_\xi^\beta \left(f - \sum_{j<N} f_{k-j} \right)(x, \xi) \right| \leq C \|\xi\|_H^{k-N-\langle\beta\rangle}$$
is called the *Heisenberg symbol class of order k*. Like for $S_k(U)$, we will always write $S^k(U)$ with the argument U to avoid confusion with the invariant symbols.

5. An operator $P\colon C_c^\infty(U) \to C^\infty(U)$ of the form
$$Pf(x) = p(x, -iX)f(x) = (2\pi)^{-d-1} \int e^{i\langle x,\xi\rangle} p(x, \sigma(x, \xi)) \hat{f}(\xi) d\xi \tag{4.25}$$
with $p \in S^k(U)$ is called a *Heisenberg pseudodifferential operator of order k* and p is called its symbol. The class of Heisenberg pseudodifferential operators of order k is denoted by $\Psi_H^k(U)$.

Remark. Our definition of the symbol differs from that of [BG88] (and agrees with that of [Pon08]) who call the function $q(x, \xi) = p(x, \sigma(x, \xi))$ the symbol.

Remark (Differential operators)**.** Consider a (local) differential operator of Heisenberg order k as defined in (4.9). Then, P has a symbol p that is the (finite) sum of homogeneous (in the Heisenberg sense) polynomials. Such a symbol is obviously in $S^k(U)$ and thus, the differential operators of Heisenberg order k on U are contained in $\Psi_H^k(U)$.

Remark. The class of Heisenberg pseudodifferential operators $\Psi_H^k(U)$ is contained in the class of pseudodifferential operators $\Psi^k_{(\frac{1}{2},\frac{1}{2})}$ as defined by Hörmander. These are characterised by a symbol satisfying

$$|D_\xi^\gamma D_x^\beta a(x,\xi)| \leq C_{\gamma,\beta}(1+\|\xi\|)^{k-\frac{|\beta|}{2}+\frac{|\gamma|}{2}}.$$

This class of operators has two major disadvantages that will be overcome by restricting ourselves to the smaller class $\Psi_H^k(U)$: It is not stable under composition and change of variables. The latter implies in particular that these operators cannot be defined on manifolds.

The order of a pseudodifferential operator can be negative, so that (heuristically speaking) while a differential operator makes a function "less smooth", a pseudodifferential operator of negative order makes it more smooth (this can be made more explicit by showing that pseudodifferential operators extend to certain Sobolev spaces, compare Proposition 4.5.6). Taking this to an extreme, we obtain (infinitely) smoothing operators.

Definition. A continuous linear operator $T\colon C_c^\infty(U) \to C^\infty(U)$ is called a *smoothing operator* if it extends to a continuous linear map

$$T\colon \mathcal{E}'(U) \to C^\infty(U),$$

where $\mathcal{E}'$ is the topological dual of $\mathcal{E}(U) = C^\infty(U)$ (the space of distributions with compact support), compare Appendix A.1.

Proposition 4.4.12 ([BG88, Prop. 10.45])**.** *Let $P \in \Psi_H^k(U)$ for some $k \in \mathbb{Z}$. Then, P is smoothing if and only if it is in $\Psi_H^{-\infty}(U) = \bigcap_{j\in\mathbb{N}} \Psi_H^{-j}(U)$.*

Almost any sequence of homogeneous functions can be made into the symbol of a pseudodifferential operator, as the following lemma shows:

Lemma 4.4.13 ([BG88, Proposition 10.10])**.** *Let $p_j \in S_j(U)$ for $j = k, k-1, \ldots$. Then there exists a symbol $p \in S^k(U)$ with asymptotic expansion*

$$p \sim \sum_{j=0}^{\infty} p_{k-j}.$$

Any two operators with this asymptotic expansion differ by a smoothing operator.

Using the same arguments as for Lemma 4.4.2, we obtain the following result.

Lemma 4.4.14. *The asymptotic expansion of any $p \sim \sum p_{k-j} \in S^k(U)$ is unique.*

This justifies the following definition:

Definition. Let $p \sim \sum_{j=0}^{\infty} p_{k-j} \in S^k(U)$, then p_k is called the *principal symbol* of the associated operator.

Note that our definition of the class of Heisenberg symbols depends on the choice of H-frame $X_0, \ldots, X_d$. And while the asymptotic expansion of a symbol may indeed change with a change of H-frame, the class of pseudodifferential operators does not.

Proposition 4.4.15 ([BG88, Prop 10.46])**.** *The class $\Psi_H^m(U)$ of Heisenberg pseudodifferential operators does not change when we replace $X_0, \ldots, X_d$ by a different H-frame $\tilde{X}_0, ..., \tilde{X}_d$.*

Kernels of pseudodifferential operators

We have essentially discussed kernels in the section on left-invariant pseudodifferential operators. All we need to do now is add the dependence on the point $x \in U$.

Definition.

1. The class $\mathcal{K}_k(U)$ consists of distributions $K \in C^\infty(U)\hat{\otimes}\mathcal{S}'_\infty(\mathbb{R}^{d+1})$ satisfying
$$K_\lambda = \lambda^k K + (\lambda^k \ln \lambda) a,$$
where the dilations act in the second variable and a is smooth in the first variable and a (Heisenberg-)homogeneous polynomial of order k in the second variable.

2. The class $\mathcal{K}^k(U)$ consists of distributions $K \in \mathcal{D}'(U \times \mathbb{R}^{d+1})$ with an asymptotic expansion
$$K \sim \sum_{j=0}^{\infty} K_{k+j}, \quad K_{k+j} \in \mathcal{K}_{k+j},$$
in the sense that for every $N > 0$ there exists some $J > 0$ such that
$$K - \sum_{j=0}^{J} K_{k+j} \in C^N(U \times \mathbb{R}^{d+1}).$$

We then have the following characterisation of Heisenberg pseudodifferential operators via their kernels.

Proposition 4.4.16 ([Pon08, Prop 3.1.15])**.** *Let $P \colon C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator with distributional kernel K_P. Then the following statements are equivalent:*

(1°) P is a Heisenberg pseudodifferential operator of order k with symbol $p \in S^k(U)$.

(2°) We can write the kernel in the following form:

$$K_P(x,y) = \det(d\psi_x) K_P^A(x, -\psi_x(y)) + R(x,y), \tag{4.26}$$

where $K_P^A \in \mathcal{K}^{-k-d-2}$, ψ_x is the adapted coordinate map at x and $R \in C^\infty(U \times \mathbb{R}^{d+1})$.

Moreover, the asymptotic expansions of kernel and symbol agree in the following sense: for $K_P^A \sim \sum K_{-k-d-2+j}$ and $p \sim \sum p_{k-j}$, we have that p_{k-j} is the restriction to $\mathbb{R}^{d+1} \setminus \{0\}$ of $\mathcal{F}_{y\to\xi}(K_{-k-d-2+j})$, where $\mathcal{F}_{y\to\xi}$ denotes Fourier transform in the y-variable.

In Heisenberg coordinates, this may be rewritten as follows.

Proposition 4.4.17 ([Pon08, Prop 3.1.16, Rem 3.1.17])**.** *Let $P\colon C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator with distributional kernel K_P. Then the following statements are equivalent:*

(1°) P is a Heisenberg pseudodifferential operator of order k with symbol $p \in S^k(U)$.

(2°) We can write the kernel in the form

$$K_P(x,y) = \det(d\varepsilon_x) K_P^H(x, -\varepsilon_x(y)) + R(x,y), \tag{4.27}$$

where $K_P^H \in \mathcal{K}^{-k-d-2}$, ε_x is the Heisenberg coordinate map at x and $R \in C^\infty(U \times \mathbb{R}^{d+1})$.

Moreover, the asymptotic expansions of kernel and symbol agree in the following sense: for $K_P^H \sim \sum K_{-k-d-2+j}$ and $p \sim \sum p_{k-j}$, we have that p_{k-j} is the restriction to $\mathbb{R}^{d+1} \setminus \{0\}$ of $\mathcal{F}_{y\to\xi}[K_{-k-d-2+j}(x, \phi_x^{-1}(y))]$, where ϕ_x is the isomorphism (4.8).

Moreover, with $K_P^H \sim \sum K_{-k-d-2+j}$ as above, the principal symbol $p_m(0,\xi)$ of P at $x = 0$ in Heisenberg coordinates centred at x_0 is given by

$$p_k(0,\xi) = \mathcal{F}_{y\to\xi}[K_{-k-d-2}](x_0, \xi).$$

One may state and prove an essential property of Heisenberg pseudodifferential operators in terms of kernels, namely their invariance under suitable coordinate change, a property that (together with the product structure) sets them apart from the larger class of pseudodifferential operators of class $(\frac{1}{2}, \frac{1}{2})$ in which they are contained. Obviously, we will need to use coordinates adapted to the Heisenberg structure if we want to be able to conserve the properties of the Heisenberg operators, which rely on this special structure.

Let (U, H) be as in situation 4.4.11. By a *Heisenberg change of coordinates* we mean a diffeomorphism $F\colon U \to \tilde{U}$ and a distribution $\tilde{H} \subset T\tilde{U}$ that again satisfies situation 4.4.11 such that $dF(H) = \tilde{H}$, i.e. a Heisenberg isomorphism $F\colon (U,H) \to (\tilde{U},\tilde{H})$. Recall that the Heisenberg differential of F is defined by $D_H F_a([X_0]_{H_a} + X) = \bar{dF}([X_0]_{H_a}) + dF_a(X)$ for $X \in H_a$. The proof of the Theorem (see [Pon08, Appendix A]) relies on the approximation of an Heisenberg isomorphism in Heisenberg coordinates by the Heisenberg differential (see Proposition 4.2.8).

Proposition 4.4.18 ([Pon08, Prop 3.1.18])**.** *Let (U,H) together with an H-frame $(X_0,\dots,X_d)$ be as in situation 4.4.11 and $F\colon U \to \tilde{U}$ a change of Heisenberg coordinates to $(\tilde{U},\tilde{H})$ together with an H-frame $(\tilde{X}_0,\dots,\tilde{X}_d)$. Let $\tilde{P} \in \Psi^k_{\tilde{H}}(\tilde{U})$ a Heisenberg pseudodifferential operator. Then:*

1. *The pullback of $\tilde{P}$ with respect to F is a Heisenberg pseudodifferential operator: $P = F^*\tilde{P} \in \Psi^k_H(U)$.*
2. *Let the kernel of $\tilde{P}$ in Heisenberg coordinates be given by (4.27). Then, the kernel of P in Heisenberg coordinates is given by (4.27) with*

$$K^H_P(x,y) = \det(D_H F_a) K^H_{\tilde{P}}(F(x), D_H F_x(y)) \quad \operatorname{mod} \mathcal{K}^{-k-d-1}(U).$$

Remark. Note that the image of Heisenberg coordinates ε_x lives in $T_H M_x$ in standard coordinates. Therefore, $D_H F_x(y)$ makes sense in the above formula.

This invariance under Heisenberg coordinate changes will allow us to define Heisenberg pseudodifferential operators on general Heisenberg manifolds.

Finally, one may also use kernels to show that the transposes $P^t\colon \mathcal{E}'(U) \to \mathcal{D}'(U)$ of an Heisenberg pseudodifferential operator is again an Heisenberg pseudodifferential operator. Recall that P^t is defined by $(P^t L)(f) = L(Pf)$, compare Appendix A.1.

Proposition 4.4.19 ([Pon08, Prop 3.1.21])**.** *Let $P \in \Psi^k_H(U)$. Then:*

1. *The transpose operator restricts to an operator $P^t\colon C^\infty_c(U) \to C^\infty(U)$ and is a Heisenberg pseudodifferential operator of order k.*
2. *If we write the distributional kernels of P and P^t in the form (4.27), then*

$$K^H_{P^t}(x,y) = K^H_P(x,-y) \quad \operatorname{mod} \mathcal{K}^{-k-d-1}.$$

Products of Heisenberg pseudodifferential operators

As for the invariant operators, we want to be able to compose two Heisenberg ΨDOs. The multiplications structure of the symbols will then be given pointwise, by the product for invariant operators. To see that this defines a smooth symbol again, we need to ensure that this depends smoothly on the point.

Proposition 4.4.20 ([BG88, Prop 13.3, 13.9])**.** *Let $p_j \in S_{k_j}(U)$. Then, $(p_1 \# p_2)(a, \cdot) = p_1(a, \cdot)\#_a p_2(a, \cdot)$ is smooth in a. The product $\#$ thus defined is a well-defined map*

$$\#\colon S_{k_1}(U) \times S_{k_2}(U) \longrightarrow S_{k_1+k_2}(U).$$

Before we go on stating a result on the composition of Heisenberg ΨDOs, note that we defined Heisenberg ΨDOs as operators from $C_c^\infty(U)$ to $C^\infty(U)$ so that the composition is not a priori well-defined. This problem is circumvented if one of the operators is properly supported.

Definition. A linear operator $T\colon C_c^\infty(U) \to C^\infty(U)$ is called *properly supported* if $T(C_c^\infty(U)) \subset C_c^\infty(U)$ and T extends to a continuous operator $T\colon C^\infty(U) \to C^\infty(U)$.

Up to a change by a smoothing operator, pseudodifferential operators are always properly supported.

Proposition 4.4.21 ([BG88, Prop 9.25, Remark 10.79])**.** *Assume $p \in S^k(U)$. Then there exists a symbol $\hat{p} \in S^k(U)$ such that $\hat{p}(x, -iX)$ is properly supported and $p - \hat{p} \in S^{-\infty}(U)$.*

Now, knowing that any ΨDO can be made into a properly supported one, we can state the following result:

Proposition 4.4.22 ([BG88, Theorem 14.1])**.** *Suppose $P_j \in \Psi_H^{k_j}(U)$ are properly supported. Then, $P = P_1 P_2 \in \Psi_H^k(U)$ where $k = k_1 + k_2$ and the principal symbol p_k is given by the product of the two principal symbols: $p_k = (p_1)_{k_1} \# (p_2)_{k_2}$.*

This product structure is in fact one of the two the main reason why Heisenberg ΨDOs were considered in the first place (the other reason is their invariance under appropriate coordinate changes discussed before). As mentioned before, any element of $\Psi_H^k(U)$ is also a ΨDO of type $(\frac{1}{2}, \frac{1}{2})$ as introduced by Hörmander. However, the product of two such operators is, in general, not a pseudodifferential operator of any reasonable class anymore.

In fact, we cannot only determine the principal symbol of the composed operator P, but the full asymptotic expansion of its symbol. The interested reader can find these asymptotics in [BG88, Thm 14.7].

4.4.3 Heisenberg pseudodifferential operators on Heisenberg manifolds and their symbols

The fact that the kernels of Heisenberg ΨDOs behave well under a change of coordinates allows us to define Heisenberg ΨDOs on a Heisenberg manifold.

Definition. The class of *Heisenberg pseudodifferential operators* of order $k \in \mathbb{Z}$ on a vector bundle E over a Heisenberg manifold consists of linear continuous operators $P\colon \Gamma_c(E) \to \Gamma(E)$ such that

(i) The distributional kernel of P is smooth off the diagonal.

(ii) For any open sets $U \subset M$, $V \subset \mathbb{R}^{d+1}$ and any trivialisation $\tau\colon E|_U \to V \times \mathbb{C}^r$, the operator $\tau_*(P|_U)$ belongs to $\Psi_H^k(V) \otimes \mathrm{End}(\mathbb{C}^r)$, i.e. it is an $r \times r$-system of Heisenberg pseudodifferential operators of order k.

We denote this class by $\Psi_H^k(M, E)$.

Remark (Differential operators)**.** A differential operator of Heisenberg order k is defined by its local form. Thus, it follows from the discussion in the last section that the class of differential operators of Heisenberg order k is contained in $\Psi_H^k(M, E)$.

Lemma 4.4.23 ([BG88, Remark (19.2)])**.** *Let $P \in \Psi_H^k(M, E)$. Then P extends to a continuous linear operator*

$$\mathcal{E}'(M, E) \to \mathcal{D}'(M, E).$$

For the definition of the space of generalised sections $\mathcal{E}'(M, E), \mathcal{D}'(M, E)$, compare Appendix A.1. The properties of Heisenberg pseudodifferential operators discussed in the previous section have obvious extensions to systems of operators. Componentwise, one may consider their kernels, giving system of kernels. The product of two such operators is given via the products of the components and the product structure for symbols carries over in the same way. This carries over to operators on manifolds and gives

Proposition 4.4.24. *Let $P_j \in \Psi_H^{k_j}(M, E)$. Then the composition $P_1 P_2$ is a Heisenberg pseudodifferential operator of order $k_1 + k_2$ on M.*

We will discuss the products in more detail later, including their principal symbol (which we have yet to define). Before we do so, we discuss transpose and adjoint operators. For $P \in \Psi_H^k(M, E)$, the transpose operator $P^t\colon \Gamma(E)' \to \Gamma_c(E)'$ is defined by $(PL)(s) = L(Ps)$ for any section $s \in \Gamma_c(E)$, compare Appendix A.1. Using the local result that transpose operators are again H-pseudodifferential operators, we obtain the global analogue.

Proposition 4.4.25 ([Pon08, Prop 3.1.23])**.** *Let $P\colon \Gamma_c(E) \to \Gamma(E)$ be a $H - \Psi DO$ of order k. Then, the transpose operator induces an operator*

$$P^t\colon \Gamma_c(E^* \otimes |\Lambda(M)|) \to \Gamma(E^* \otimes |\Lambda(M)|)$$

that is again a $\Psi_H DO$ of order k, where $|\Lambda(M)|$ is the density bundle of M.

If we assume M to be equipped with a positive density and a Hermitian metric on E, we can form the formal adjoint operator $P^*\colon \Gamma_c(E) \to \Gamma(E)$, defined by $(P^*e, f)_{L^2} = (e, Pf)_{L^2}$ for any $e, f \in \Gamma_c(E)$. The adjoint P^* is then again a $\Psi_H DO$ of order k.

Principal symbols of Heisenberg operators

In this section, we define the global principal symbol of a Heisenberg operator on a Heisenberg manifold M. Like for differential operators, we can simply define it as the image of P under the quotient map

$$\Psi_H^k(M,E) \to \Psi_H^k(M,E)/\Psi_H^{k-1}(M,E).$$

We will now give a more concrete description of the Heisenberg principal symbol by identifying the quotient space with a suitable space of functions.

Definition. Denote $\pi\colon \mathfrak{t}_H\mathfrak{m}^* \to M$ the projection of the bundle of co-tangent Lie algebras. The symbol class $S_k(M,E)$ consists of sections $p \in \Gamma(\mathfrak{t}_H\mathfrak{m}^*, \mathrm{End}(\pi^*E))$ which are homogeneous of degree k in the second variable, i.e.

$$p(x, \lambda.\xi) = \lambda^k p(x,\xi) \quad \text{for all } \lambda > 0,$$

where $\lambda.(\xi_0 + \xi_H) = \lambda^2\xi_0 + \lambda\xi_H$ for $\xi = \xi_0 + \xi_H \in \mathfrak{t}_H\mathfrak{m}^* \simeq (TM/H)^* \oplus H^*$.

Then, the principal symbol can be understood as an element of this space.

Theorem 4.4.26 ([Pon08, Thm 3.2.2, Prop 3.2.6])**.** *For any $P \in \Psi_H^k(M,E)$, there exists a unique element $\sigma_H^k(P) \in S_k(M,E)$ with the following property: In a local trivialising chart $\kappa : V \to U$, the leading part $K_{-k-d-2} \in \mathcal{K}_{-k-d-2}(U)$ of the kernel K_P^H of $\kappa_* P$ according to (4.27) is related to the symbol $\sigma_H^k(P)$ as follows.*

$$\sigma_H^k(P)(x,\xi) = \mathcal{F}_{y\to\xi}[K_{-k-d-2}](x,\xi). \tag{4.28}$$

Equivalently, $\sigma_H^k(P)(x_0,\cdot)$ is the principal symbol (in the local sense, as the highest-order part of the expansion) of P at $x = 0$ in trivialising Heisenberg coordinates.

The map $\sigma_H^k\colon \Psi_H^k(M,E) \to S_k(M,E)$ gives rise to an isomorphism

$$\sigma_H^k\colon \Big(\Psi_H^k(M,E)/\Psi_H^{k-1}(M,E)\Big) \xrightarrow{\simeq} S_k(M,E).$$

Definition. We will call $\sigma_H^k(P)$ as defined in Theorem 4.4.26 the (global) *Heisenberg principal symbol* (or H-principal symbol) of P.

Proof. The full proof may be found in the quoted monograph by Ponge. We reproduce here the proof of the essential fact that the symbol is well-defined.

We need to show that $\sigma_H^k(P)$ is well-defined by (4.28), i.e. that this is independent of the choice of coordinates. We can reduce this to the case of scalar operators, because in a local trivialisation, a bundle-valued operator becomes a system of scalar operators and the following arguments can be carried out for each scalar operator.

Let $\kappa_j \colon V_j \to W_j$ be local trivialising charts and $U_j = \kappa_j(V_1 \cap V_2)$. We consider the operators $P_j = (\kappa_j)_* P$ and the transition map $F \colon U_1 \to U_2$. We put the distributional kernels of P_1 and P_2 in the form (4.27) with

$$K_{P_1}^H \sim \sum K^1_{-k-d-2+j} \in \mathcal{K}^{-k-d-2}, \quad K_{P_2}^H \sim \sum K^2_{-k-d-2+j} \in \mathcal{K}^{-k-d-2}.$$

Then, because $P_1 = F^* P_2$, the invariance theorem 4.4.18 yields that

$$K_{P_1}^H(x,y) = \det(D_H F_x) K_{P_2}^H(F(x), D_H F_x(y)) \mod \mathcal{K}^{-k-d-1}$$

and in particular

$$K^1_{-k-d-2}(x,y) = \det(D_H F_x) K^2_{-k-d-2}(F(x), D_H F_x(y)).$$

We set

$$p_k^j(x,\xi) := \mathcal{F}_{y\to\xi}[K^j_{-k-d-2}](x,\xi).$$

By the last statement of proposition 4.4.17, $p_k^j(x,\cdot)$ is the principal symbol of P_j at 0 in Heisenberg coordinates centred at x. Because $D_H F_x$ is linear, it transforms under the Fourier transform to give

$$p_k^1(x,\xi) = p_k^2(F(x), (D_H F_x^{-1})^T \xi).$$

Thus $p_k := \kappa_1^* p_k^1$ is independent of the choice of coordinates. □

Remark (Global and local principal symbol, cf [Pon08, Rem 3.2.4])**.** We will call the principal symbol defined above the global principal symbol and p_k according to section 4.4.2 the local principal symbol should we need to distinguish between them. In suitably centred Heisenberg coordinates, the two symbols agree. We want to compare the two symbols for general coordinates. Using proposition 4.4.17, we have that (notation as in the proposition)

$$p_k(x,\xi) = \mathcal{F}_{y\to\xi}[K_{-k-d-2+j}(x, \phi_x^{-1}(y))],$$

where $\phi_x \colon G^{(x)} \to T_H M_x$ is the isomorphism (4.8). Thus, by the definition of the global principal symbol

$$p_k(x,\xi) = \mathcal{F}_{y\to\xi}\left[\mathcal{F}^{-1}_{\xi\to y}[\sigma_k(P)](x, \phi_x^{-1}(y))\right](x,\xi).$$

In what follows, we will denote the right-hand side above by $\hat{\phi}_x^* \sigma_k(P)$.

Remark (Differential operators)**.** To begin with, recall that the elements of the enveloping algebra of degree k, considered as (polynomial) functions, are elements of $S_k(M,E)$, compare the remark at the end of section 4.3. In a local H-frame $(X_0,\dots,X_d)$, assume that a differential operator P has the local form

$$(Pf)(x) = p(x,-iX)f(x) = \sum_{\langle\gamma\rangle\le k} b_\gamma(x)((-iX)^\gamma f)(x).$$

We will still have this form in Heisenberg coordinates. Then, the local principal symbol is given by $p_k(x,\xi) = \sum_{\langle\gamma\rangle=k} b_\gamma(x)\xi^\gamma$. Using that the local principal symbol in suitably centred Heisenberg coordinates agrees with the global principal symbol, we obtain that the global principal symbol has the form

$$\sigma_H^k(P)(x,\xi) = \sum_{\langle\gamma\rangle=k} b_\gamma(x)\xi^\gamma.$$

Like for differential operators, each pseudodifferential operator has a model operator at each point.

Definition. Let $P \in \Psi^k(M,E)$. Then, the *model operator* of P at a is the left-invariant homogeneous pseudodifferential operator

$$P^a\colon \mathcal{S}_0(T_HM_a,E_a) \to \mathcal{S}_0(T_HM_a,E_a)$$

with symbol $\sigma_H^k(P)(a,\cdot)$.

Composition of symbols

We will again need a product of symbols that gives the symbol of the composition of operators. In particular, the composition of symbols will give rise to the notion of invertible symbol which will turn out to be equivalent to the existence of a parametrix.

We begin by noting that the symbol products of sections 4.4.1 and 4.4.2 extend to systems of operators in the obvious way. Now, for any $\sigma \in S_k(M,E)$ and any $a \in M$, the symbol $\sigma(a,\cdot)$ is in $S_k \otimes \mathrm{End}(E_a)$. Thus, we can define a product of two symbols $\sigma_1 \in S_k(M,E)$ and $\sigma_2 \in S_l(M,E)$ as follows.

$$(\sigma_1 \# \sigma_2)(a,\xi) = (\sigma_1(a,\cdot)\#_a\sigma_2(a,\cdot))(\xi). \tag{4.29}$$

Pointwise, the result is in S_{k+l}. By [BG88, Prop 13.3], this depends smoothly on the point a (the result in [BG88] is for opens subsets of $\mathbb{R}^n$, but smoothness is a local property). Thus, we obtain

Proposition 4.4.27 ([Pon08, Proposition 3.2.8]). *Equation* (4.29) *gives rise to a well-defined product map*

$$\#\colon S_k(M,E)\times S_l(M,E)\longrightarrow S_{k+l}(M,E).$$

Like for the local symbols, this is the correct definition of the product of symbols in the sense that the product of symbols yields the symbol of products.

Proposition 4.4.28 ([Pon08, Prop 3.2.9]). *Let $P_j\in\Psi_H^{k_j}(M,E)$ $(j=1,2)$ and assume one of them to be properly supported. Then, $P_1P_2\in\Psi_H^{k_1+k_2}(M,E)$ and*

1. *the principal symbol of the composition is given by the product of symbols: $\sigma_H^{k_1+k_2}(P_1P_2)=\sigma_H^{k_1}(P_1)\#\sigma_H^{k_2}(P_2)$.*
2. *at any $a\in M$, the model operator of the product is given by the composition of model operators: $(P_1P_2)^a=P_1^aP_2^a$.*

Proof. For the first property, one uses the comparison between local and global symbol and the corresponding properties of the local product (the interested reader will find the details in [Pon08]). The second property then follows using the properties of the product of homogeneous symbols. □

One may additionally show that the product of symbols is a continuous bilinear map for a suitable topology on the symbol spaces (cf [Pon08, Prop 3.2.10]).

Principal symbols of transposes and adjoints

The results on the symbols of transposes and adjoints for local operators carry over to operators on manifolds and we obtain similar results for model operators.

Proposition 4.4.29 ([Pon08, Prop 3.2.11, 3.2.12]). *Let $P\in\Psi_H^k(M,E)$ have principal symbol $\sigma_H^k(P)$. Then the principal symbol of the transpose and formal adjoint may be calculated as follows:*

1. *The principal symbol of P^t is given by $\sigma_H^k(P^t)(x,\xi)=\sigma_H^k(P)(x,-\xi)^t\otimes id_{|\Lambda(M)|}$.*

   The model operator of the transpose operator is the transpose of the model operator: $(P^t)^a=(P^a)^t\colon\mathcal{S}_0(T_HM_a,E_a^*)\to\mathcal{S}_0(T_HM_a,E_a^*)$ (the tensor product with the density bundle vanishes on trivial bundles).
2. *If M is endowed with a smooth density dx and E carries a Hermitian metric, the principal symbol of the formal adjoint operator P^* is given by $\sigma_H^k(P^*)(x,\xi)=\sigma_H^k(P)(x,\xi)^*$.*

   The model operator of the formal adjoint operator is the formal adjoint of the model operator: $(P^*)^a=(P^a)^*\colon\mathcal{S}_0(T_HM_a,E_a)\to\mathcal{S}_0(T_HM_a,E_a)$

4.5 Hypoellipticity of H-differential operators

We now come to the "heart" of the Heisenberg calculus: We obtain conditions for the hypoellipticity of Heisenberg differential operators that may be (fairly) easily checked. The key ingredient of the criterion is that the hypoellipticity of the operator P depends on the hypoellipticity of its model operators P^a at each point. The hypoellipticity of P^a may then be checked through a representation-theoretic criterion, the so-called *Rockland condition.* This condition was first proven by C. Rockland for homogeneous differential operators on the Heisenberg group, cf [Roc78]. It was extended to general nilpotent graded groups by Helffer and Nourrigat ([HN79]) and to pseudodifferential operators by P. Głowacki [Gło89, Gło91]. The papers cited here are just a few among many publications on the subject.

We begin by discussing some background information on the Rockland condition for homogeneous differential operators on groups. We then move on to using this condition to determine the hypoellipticity of H-pseudodifferential operators on Heisenberg manifolds. Next, we show that the spectrum of symmetric hypoelliptic Heisenberg differential operators mirrors many properties of that of elliptic operators. We then apply this theory to Heisenberg differential operators of order two and, in particular, to the squares of horizontal Dirac operators. In a final section, we discuss partial inverses to treat the parts of the horizontal Dirac operator that are not hypoelliptic.

4.5.1 Rockland operators

In this section, assume that G is a graded nilpotent group and V any vector space. The theory of Rockland operators relies on representations of G. We fix the notation: We consider all unitary representations

$$\pi\colon G \to U(\mathcal{H}_\pi)$$

of the group g. Recall that a unitary operator on a Hilbert space H is a bounded operator $T\colon H \to H$ satisfying $TT^* = T^*T = Id$, where T^* is the adjoint. The set of all equivalence classes of unitary representations is denoted $\hat{G}$ and called the unitary dual of G. For any $\pi \in \hat{G}$, let

$$C^\infty(\pi) = \{\, u \in \mathcal{H}_\pi \mid \ g \mapsto \pi(g)u \text{ is smooth}\}$$

denote the smooth elements of π.

The case of scalar differential operators

Any representation $\pi \in \hat{G}$ induces a representation π_* of the Lie algebra on $\mathcal{H}_\pi$. This induced representation maps any $X \in \mathfrak{g}$ to an unbounded operator $\pi_*(X)\colon C^\infty(\pi) \subset$

$\mathcal{H}_\pi \to \mathcal{H}_\pi$ defined by

$$\pi_*(X)(u) = \frac{d}{dt}\pi(exp(tX))u|_{t=0}.$$

Then, π_* extends to the enveloping algebra of $\mathfrak{g}$ in the obvious way, which allows us to consider $\pi_*(L)$ for invariant differential operators on G. These representations of operators will give a criterion for the hypoellipticity of the operator.

Definition. We say that a homogeneous (in the graded sense) differential operator

$$L\colon C^\infty(G) \to C^\infty(G)$$

that is left-invariant satisfies the *Rockland condition* if $\pi_*(L)\colon C^\infty(\pi) \to \mathcal{H}_\pi$ is injective for every nontrivial $\pi \in \hat{G}$. An operator that satisfies the Rockland condition is also called a Rockland operator.

Remark ([vE05, Lemma 41]). Rockland operators are a generalisation of elliptic operators in the following sense:

Let $G = \mathbb{R}^n$ be the abelian group with trivial grading and L an invariant (i.e. constant coefficients) homogeneous differential operator on $\mathbb{R}^n$. Then P is a Rockland operator if and only if it is elliptic.

This is due to the fact that representations of $\mathbb{R}^n$ are scalar and of the form $\pi_\xi(x) = e^{i\langle x,\xi\rangle}$. with $\xi \in (\mathbb{R}^n)^* \simeq \mathbb{R}^n$. Then, $(\pi_\xi)_*(L)$ is simply the symbol of L in the usual sense and it is injective if and only if it is invertible.

We then have the following result, which first appeared in [Roc78] for the Heisenberg group. Rockland already conjectured that this result should hold for more general groups. Beals [Bea77] proved necessity in the general case and sufficiency for products of the Heisenberg group with $\mathbb{R}^k$. Sufficiency in the general case was proven by Helffer and Nourrigat [HN79].

Theorem 4.5.1. *A left-invariant homogeneous differential operator on a graded group is hypoelliptic if and only if it satisfies the Rockland condition.*

The case of systems

We now consider invariant operators acting on sections of a trivial vector bundle $G \times V \simeq G \times \mathbb{C}^r$. We content ourselves with trivial bundles here because this is the situation that we are faced with when discussing model operators. Equivalently, we may consider such an operator L as a system (or a matrix) of operators $L_{\mu\nu}\colon C^\infty(G) \to C^\infty(G)$. Now, a representation will act on such an operator as follows: $\pi(L)$ will be a linear operator on $\mathcal{H}_\pi^{\oplus r} = \underbrace{\mathcal{H}_\pi \oplus ... \oplus \mathcal{H}_\pi}_{r}$ with domain $(C^\infty(\pi))^{\oplus r}$

defined by $\pi(L) = (\pi(L_{\mu\nu}))_{\mu\nu}$, i.e.

$$\pi(L)(v_1, ..., v_r) = \left(\sum_{\nu} \pi(L_{1\nu})(v_\nu), ..., \sum_{\nu} \pi(L_{r\nu})(v_\nu) \right).$$

Again, we have the notion of Rockland operator.

Definition. A homogeneous (in the graded sense) differential operator

$$L\colon C^\infty(G, \mathbb{C}^r) \to C^\infty(G, \mathbb{C}^r)$$

that is left-invariant satisfies the *Rockland condition* if $\pi_*(L)\colon (C^\infty(\pi))^{\oplus r} \to \mathcal{H}_\pi^{\oplus r}$ is injective for every nontrivial $\pi \in \hat{G}$.

Theorem 4.5.1 can then be extended to systems of operators.

Theorem 4.5.2. *A left-invariant homogeneous differential operator acting on a trivial vector bundle over a graded group is hypoelliptic if and only if it satisfies the Rockland condition.*

The sufficiency of the Rockland condition has been proven by van Erp [vE05, Appendix A], essentially following [HN79] and making adjustments where necessary. The necessity can be shown by adapting the proof of Beals [Bea77] for scalar operators.

4.5.2 Hypoellipticity of Heisenberg differential operators

In this section, we want to give a criterion for the existence of a parametrix (and thus a sufficient condition for hypoellipticity) for Heisenberg differential operators via the Rockland condition for the model operator at every point. We begin with the following result on the relation between parametrices and the principal symbol.

Proposition 4.5.3 ([Pon08, Prop 3.3.1])**.** *Let $P\colon \Gamma_c(E) \to \Gamma(E)$ be a differential operator of Heisenberg order k. Then P admits a parametrix $Q \in \Psi_H^{-k}(M, E)$, i.e. $PQ = QP = I \bmod \Psi^{-\infty}(M, E)$, if and only if its Heisenberg principal symbol $\sigma_H^k(P)$ is invertible in $S_*(M, E)$ with respect to the product* (4.29)*.*

One might be tempted to think that the invertibility of the symbol is a condition that can checked pointwise as the product of symbols is defined pointwise. However, bear in mind that doing so does not guarantee that the resulting "symbol" depends smoothly on the point.

As the existence of a parametrix will guarantee "nice" analytic properties of the operator, we make the following definition.

Definition. We call the differential operator $P\colon \Gamma_c(E) \to \Gamma(E)$ of Heisenberg order k *Heisenberg-elliptic* (or H-elliptic) if its Heisenberg principal symbol is invertible.

The invertibility of the H-principal symbol of P is equivalent to the Rockland condition for P^a at each point.

Theorem 4.5.4 ([Pon08, Thm 3.3.10, 3.3.18])**.** *Let $P\colon \Gamma_c(E) \to \Gamma(E)$ be a differential operator of Heisenberg order k. Then, the following statements are equivalent:*

(1°) P is Heisenberg-elliptic.

(2°) The model operators P^a and $(P^a)^t = (P^t)^a$ satisfy the Rockland condition at every point $a \in M$.

If $k = 0$ either statement is equivalent to

(3°) At each point $a \in M$, P^a is invertible on $L^2(T_H M_a, E_a)$.

If E is equipped with a Hermitian scalar product, $(P^t)^a$ may be replaced by $(P^*)^a$.

Proof. We begin by proving the case $k = 0$.

(1°) $\Rightarrow$ (2°) If the symbol of P is invertible, P admits a parametrix $Q \in \Psi_H^{-k}(M, E)$. Then Q^a and $(Q^a)^t$ are parametrices for P^a and $(P^a)^t$ respectively and thus, P^a and $(P^a)^t$ are hypoelliptic and therefore satisfy the Rockland condition.

(2°) $\Leftrightarrow$ (3°) An invariant differential operator of order zero is simply a matrix A. Going over to the representation $\pi_*(A)$ only lets the matrix act on a different space and thus, A is invertible if and only if $\pi_*(A)$ is invertible, which is equivalent to the left-invertibility (and thus injectivity) of $\pi_*(A)$ and $\pi_*(A^t)$.

(3°) $\Rightarrow$ (1°) This is proven in [Pon08, Thm 3.3.10].

The general case $k \in \mathbb{Z}$ is proven by a reduction of order, see [Pon08, Thm 3.3.18]. ☐

Heisenberg-ellipticity implies hypoellipticity, i.e. we have regularity and a-priori-estimates for these operators.

Proposition 4.5.5 ([Pon08, Prop 3.3.2])**.** *Let $P\colon \Gamma_c(E) \to \Gamma(E)$ be an H-elliptic differential operator of (Heisenberg) order k. Then, P is hypoelliptic, i.e. considered as an operator on $\mathcal{E}'(M, E)$ it satisfies*

$$Pu \in \Gamma(E) \quad \Rightarrow \quad u \in \Gamma(E).$$

Moreover, if M is compact, P satisfies an a-priori-estimate*, i.e. for any differential operator A of (Heisenberg) order k there exists a constant $C > 0$ such that*

$$\|Au\|_{L^2} \leq C(\|Pu\|_{L^2} + \|u\|_{L^2}). \tag{4.30}$$

Remark. By considering Sobolev spaces, one may make the hypoellipticity statement more precise, compare the following section.

4.5.3 Spectrum of Heisenberg-elliptic differential operators

In this section we want to discuss the spectrum of Heisenberg-elliptic differential operators. More precisely, we will show that the spectrum of formally self-adjoint H-elliptic operators has the same properties as if the operator were elliptic, i.e. it is a discrete pure point spectrum on closed manifolds.

We begin by noting that Heisenberg (pseudo)differential operators extend to continuous operators on the Sobolev spaces $L^2_s(E)$. This is due to the fact Ψ^k_H is contained in $\Psi^k_{\frac{1}{2},\frac{1}{2}}$ which can be extended to Sobolev spaces as a consequence of a result by Calderon and Vaillancourt [CV72]. For the definition of the Sobolev space L^2_s, compare Appendix A.3.

Proposition 4.5.6 ([Pon08, Prop 3.1.8])**.** *Let (M, H) be a compact Heisenberg manifold and $P \in \Psi^k_H(M)$. Then, for each $s \in \mathbb{R}$, P extends to a continuous linear operator*

$$P_s \colon L^2_s(E) \longrightarrow L^s_{s-k}(E)$$

if $k \geq 0$ and

$$P_s \colon L^2_s(E) \longrightarrow L^s_{s-\frac{k}{2}}(E)$$

if $k < 0$.

We now consider the extensions of a Heisenberg-elliptic operator on Sobolev spaces. We can first refine the regularity result.

Proposition 4.5.7 ([Pon08, Prop 3.3.2])**.** *Let (M, H) be a closed (i.e. compact and without boundary) Heisenberg manifold and $P \colon \Gamma(E) \to \Gamma(E)$ Heisenberg-elliptic differential operator of (Heisenberg) order $k > 0$. Then, P satisfies the following regularity condition:*

$$Pu \in L^2_s(E) \quad \Rightarrow \quad u \in L^2_{s+\frac{k}{2}}(E).$$

Theorem 4.5.8. *Let M be a closed Riemannian manifold, E a Hermitian vector bundle over M and $P \colon \Gamma(E) \to \Gamma(E)$ a Heisenberg-elliptic operator of (Heisenberg) order k. Then, P is of usual order $\leq k$. Denote by P_s the extension of P to $L^2_s(E)$. Then,*

1. *the extension P_s is Fredholm for every $s \in \mathbb{R}$.*
2. *the kernel satisfies* $\ker P_s \subset \Gamma(E)$ *and thus does not depend on s.*
3. *the index of P_s does not depend on s and is thus given by*

$$\operatorname{index}(P) = \dim\ker(P) - \dim\ker(P^*).$$

Proof. We follow the proof of the similar theorem [Shu01, Thm 8.1] for hypoelliptic pseudodifferential operators. By Proposition 4.5.6, P extends to a continuous linear operator $P_s \colon L^2_s(E) \to L^2_{s-k}(E)$ for all $s \in \mathbb{R}$. Because P is Heisenberg-elliptic, there exists a parametrix $Q \in \Psi_H^{-k}(M,E)$, which extends to an operator Q_{s-k} on $L^2_{s-k}(E)$ by Proposition 4.5.6. Because smoothing operators are compact, P then satisfies the conditions of Proposition A.2.4.

The second claim follows from the regularity of Heisenberg-elliptic operators, see Theorem 4.5.5. By Lemma A.2.3, $\dim \operatorname{coker}(P_s) = \dim \ker(P_s)^t$. By the duality between $L^2_s(E)$ and $L_{-s}(E^*)$, $(P_s)^t = (P^*)_{-s+k}$. If P is Heisenberg-elliptic, so is P^* and thus, the kernel of $(P^*)_{-s+k}$ is smooth and independent of s. The third claim follows. □

We now define the spectrum of a pseudodifferential operator. In what follows, we assume M to be closed, i.e. compact with empty boundary. We begin by discussing the extension of an H-pseudodifferential operator to L^2. Recall that any such operator extends to distributions by lemma 4.4.23. There are three ways to extend P to an (unbounded) operator on L^2. First, one can simply consider P as an unbounded operator with domain $\Gamma(E)$. This operator is not closed and one therefore replaces it with one of the following.

Definition. The *closure* or *minimal L^2-realization* of a pseudodifferential operator $P \in \Psi_H^k(M,E)$ is the unbounded linear operator

$$\overline{P} \colon \operatorname{dom}(\overline{P}) \subset L^2(E) \longrightarrow L^2(E),$$

where the domain is defined to be all $\varphi \in L^2(E)$ for which there exists a $\psi \in L^2(E)$ such that for any sequence $(\varphi_n) \subset L^2(E)$ that converges to φ, the sequence $(P\varphi_n)$ converges to ψ. Then, one defines $\overline{P}\varphi = \psi$.

The *(maximal) L^2-realisation* of a pseudodifferential operator $P \in \Psi_H^k(M,E)$ is the unbounded linear operator

$$\mathbf{P} \colon \operatorname{dom}(\mathbf{P}) \subset L^2(E) \longrightarrow L^2(E)$$

defined as the restriction of $P \colon \mathcal{E}'(M,E) \to \mathcal{D}'(E)$ to the domain

$$\operatorname{dom}(\mathbf{P}) = \left\{ u \in L^2(E) \,\middle|\, Pu \in L^2(E) \right\}.$$

As P is continuous on distributions, we have that $\overline{P} \subset \mathbf{P}$, i.e. $\operatorname{dom}(\overline{P})$ is contained in $\operatorname{dom}(\mathbf{P})$ and the two operators agree where both are defined. Moreover, in the definition of $\overline{P}$, it is enough to require one sequence φ_n that converges to φ and for which $(P\varphi_n)$ converges. We will later see that for H-elliptic operators, the two extensions agree. By the above results, for a differential operator of order k, $L^2_k(E)$ is contained in the domain of $\mathbf{P}$.

Proposition 4.5.9 ([BG88, (19.5),(19.6)]). *Let M be closed and $P \in \Psi_H^k(M,E)$. Then its L^2-realisation $\mathbf{P}$ is closed and densely defined. If $k \leq 0$, then $\mathbf{P}$ is bounded, the adjoint of $\mathbf{P}$ is the L^2-realisation of the formal adjoint P^* of P. If $k < 0$, $\mathbf{P}$ is compact.*

Proof. The fact that operators of strictly negative order are compact is not in [BG88]. It follows from the fact that an operator of negative order is bounded as an operator $L^2(E) = L^2_0(E) \to L^2_{-\frac{k}{2}}(E)$. As $-\frac{k}{2}$ is positive, the embedding $L^2_{-\frac{k}{2}}(E) \hookrightarrow L^2(E)$ is compact. $\square$

Definition. Let $P \in \Psi_H^k(M,E)$. The *spectrum* $\operatorname{Spec}(P)$ of P consists of all $\lambda \in \mathbb{C}$ for which $(\mathbf{P} - \lambda I)$ does not have a bounded, everywhere defined inverse on $L^2(E)$.

In order to prove that formally self-adjoint H-elliptic differential operators have discrete point spectrum, we make use of the results of [BG88] on partial inverses.

Definition. Let H be a Hilbert space. Suppose that the linear operator $P\colon \operatorname{dom} P \subset H \to \operatorname{ran} P \subset H$ is closed and suppose $\operatorname{ran} P \subset H$ closed. Then we denote $\Pi_{1,2}$ the orthogonal projections

$$\Pi_1\colon H \to (\ker P)^{\perp} \quad \text{and} \quad \Pi_2\colon H \to \operatorname{ran} P.$$

By the *partial inverse* of P we mean the unique bounded operator $A\colon H \to H$ which satisfies

$$PA = \Pi_2 \quad \text{and} \quad AP = \Pi_1 \text{ on } \operatorname{dom} P.$$

The following proposition will be very useful to determine whether a given operator is indeed the partial inverse of an operator P.

Proposition 4.5.10 ([BG88, Thm 19.8]). *Let (M,H) be a closed Heisenberg manifold and let $P, \Pi_j, A\colon \mathcal{E}'(M,E) \to \mathcal{D}'(M,E)$ be Heisenberg-pseudodifferential operators and let the order of A, Π_j be ≤ 0. Suppose further that the operators satisfy the relations*

$$PA = \Pi_2 \quad \textit{and} \quad AP = \Pi_1, \tag{4.31}$$

$$\Pi_2 P = P = P\Pi_1, \tag{4.32}$$

$$\Pi_j^2 = \Pi_j = \Pi_j^*. \tag{4.33}$$

Then, the L^2-realisation $\mathbf{P}$ has closed range, $\mathbf{\Pi}_j$ are the associated projections and $\mathbf{A}$ is the associated partial inverse. Moreover, $\mathbf{P}$ is the closure of the restriction of P to $\Gamma(M,E)$ and the adjoint of the L^2-realisation $\mathbf{P}^*$ is the L^2-realisation of P^*. In particular, if P is formally self-adjoint, then $\mathbf{P}$ is self-adjoint.

Proposition 4.5.11 ([BG88, Thm 19.16])**.** *Let M be closed and let $P \in \Psi^k_H(M,E)$ have a parametrix in $\Psi^*_H(M,E)$. Then, the L^2-realisation $\mathbf{P}$ has closed range and the associated projections and partial inverse are H-pseudodifferential operators. The kernel of $\mathbf{P}$ is finite-dimensional and consists of smooth sections of E. Moreover, the parametrix and the partial inverse differ by a smoothing operator.*

Proof. The last claim is not in the statement of the theorem in [BG88], but it is present in the proof. □

Corollary 4.5.12. *Let M be closed and let $P\colon \Gamma(E) \to \Gamma(E)$ be an H-elliptic differential operator. Then, the L^2-realisation $\mathbf{P}$ has closed range and the associated projections and partial inverse are H-pseudodifferential operators. The kernel of $\mathbf{P}$ is finite-dimensional and consists of smooth sections of E. Moreover, $\mathbf{P}$ is the closure of the restriction of P to $\Gamma(M,E)$ and the adjoint of the L^2-realisation $\mathbf{P}^*$ is the L^2-realisation of P^*. In particular, if P is formally self-adjoint, then $\mathbf{P}$ is self-adjoint.*

Proof. If P is H-elliptic, it has a parametrix in $\Psi^{-k}(M,E)$ and we can thus apply the results of Proposition 4.5.11. If the partial inverse and the projections are H-pseudodifferential operators, they must satisfy (4.31)-(4.33) and we can thus use Proposition 4.5.10. □

Lemma 4.5.13. *Let $P \in \Psi^k_H(M,E)$ be formally self-adjoint and have a parametrix in $\Psi^*_H(M,E)$. Then, the spaces $L^2(E)$ admits an orthogonal direct sum decomposition*

$$L^2(E) = \ker \mathbf{P} \oplus \operatorname{ran} \mathbf{P}.$$

In particular, this holds for H-elliptic differential operators.

Proof. As $\ker \mathbf{P}$ is finite-dimensional and therefore closed, there is a decomposition $L^2(E) = \ker \mathbf{P} \oplus (\ker \mathbf{P})^\perp$. As P is formally self-adjoint, $\mathbf{P}$ is self-adjoint by Propositions 4.5.10 and 4.5.11. Thus, as $\mathrm{dom}(\mathbf{P}) \subset L^2(E)$ is dense, we have that $u \in \ker \mathbf{P}$ if and only if

$$0 = (\mathbf{P}u, v)_{L^2} = (u, \mathbf{P}v)_{L^2}$$

for all $v \in \mathrm{dom}\,\mathbf{P}$. This is equivalent to $u \perp \mathbf{P}v$. As both the kernel and the range of $\mathbf{P}$ are closed, the claim follows. □

With these preparations, we are able to prove the main result of this section.

Theorem 4.5.14. *Let (M,H) be a closed Heisenberg manifold and $P\colon \Gamma(E) \to \Gamma(E)$ a formally self-adjoint (i.e $(Pu,v) = (u,Pv)$ for all $u,v \in \Gamma(E)$) Heisenberg-elliptic differential operator of (Heisenberg) order $k > 0$. Then, $\mathbf{P}$ is self-adjoint and there exists a complete orthonormal set of eigensections $\phi_j \in \Gamma(E)$ of $\mathbf{P}$ (and thus of P). The corresponding eigenspaces are finite-dimensional, the eigenvalues are real and tend to infinity. Moreover, the spectrum of P coincides with the set of eigenvalues.*

Proof. That $\mathbf{P}$ is self-adjoint follows from Corollary 4.5.12 and thus, by Lemma 4.5.13, $\mathbf{P}(\ker \mathbf{P}^{\perp}) = \ker \mathbf{P}^{\perp}$. Let $\mathbf{A}$ be the partial inverse. On $\ker \mathbf{P}^{\perp}$, $\mathbf{A}$ is an inverse for $\mathbf{P}$. As $\mathbf{A}$ is a pseudodifferential operator of negative order, it is compact and self-adjointness carries over from $\mathbf{P}$ to its inverse $\mathbf{A}$. Thus, $\mathbf{A}|_{\ker \mathbf{P}^{\perp}}$ has pure point spectrum $(\mu_j)_j$, $|\mu_j| \to 0$ as $j \to \infty$, and an associated orthonormal system $(u_j)_j$ spanning $\ker \mathbf{P}^{\perp}$ such that $\mathbf{A}u_j = \mu_j u_j$. As $\mathbf{A}$ is self-adjoint, $\mu_j \in \mathbb{R}$. Then, the u_j are eigensections of $\mathbf{P}$ associated to the eigenvalues $\frac{1}{\mu_j}$. As $|\mu_j| \to 0$, The eigenvalues $\frac{1}{\mu_j}$ of $\mathbf{P}$ tend to infinity in absolute value. As a change of lower order does not change H-ellipticity, $P - \lambda I$ is also H-elliptic for any value λ. Thus, by Proposition 4.5.11, all eigenspaces (including the kernel) are finite-dimensional and consist of smooth sections.

The rest of the proof is very similar to the proof of Proposition 3.5.2. As the assumptions are not entirley the same ones, we reproduce the arguments here. Now, combining (u_j) with a basis of the finite-dimensional space $\ker \mathbf{P}$, we obtain an L^2-basis (v_j) of $L^2(E)$ such that $\mathbf{P}v_j = \lambda_j v_j$, where every eigenvalue appears only finitely often. We will now show that any other value λ cannot be in the spectrum. Let $u \in L^2(E)$. We define an operator $A_\lambda \colon L^2(E) \to L^2(E)$ as follows: Any $u \in L^2(E)$ can be written as $u = \sum a_j v_j$. Then, we set

$$A_\lambda u = \sum_{j=1}^{\infty} a_j \frac{1}{\lambda_j - \lambda} v_j.$$

As the eigenvalues (λ_j) do not accumulate at λ (because they tend to infinity), there exists a constant $C > 0$ such that $|\frac{1}{\lambda_j - \lambda}| \leq C$. Thus,

$$\|A_\lambda u\|_{L^2}^2 = \sum_{j=1}^{\infty} |a_j|^2 \cdot \left|\frac{1}{\lambda_j - \lambda}\right|^2 \cdot \|v_j\|_{L^2}^2 \leq C^2 \sum_{j=1}^{\infty} |a_j|^2 \cdot \|v_j\|_{L^2}^2 = C\|u\|_{L^2}^2,$$

which proves that A_λ is well-defined and bounded. We will now prove that A_λ is indeed an inverse for $\mathbf{P}_\lambda = (\mathbf{P} - \lambda \mathbf{I}) = (\mathbf{P} - \lambda I)$. The image of A_λ is in $\operatorname{dom} \mathbf{P}_\lambda$: We have

$$A_\lambda u = \lim_{n \to \infty} \sum_{j=1}^{n} a_j \frac{1}{\lambda_j - \lambda} v_j,$$

$A_\lambda u$ is the limit of a sequence in $\Gamma(E)$. The image of the sequence under $\mathbf{P}_\lambda$ converges as well, because considering P as a (continuous) operator on distributions,

$$(P - \lambda I) \sum_{j=1}^{\infty} a_j \frac{1}{\lambda_j - \lambda} v_j = \sum_{j=1}^{\infty} a_j (P - \lambda I) \frac{1}{\lambda_j - \lambda} v_j = \sum_{j=1}^{\infty} a_j v_j = u.$$

and thus, $A_\lambda u$ lies in the domain of $(\overline{P} - \lambda I)$ which is contained in (and here equal to) the domain of $\mathbf{P}_\lambda$. The last equation also shows that $\mathbf{P}_\lambda A_\lambda u = u$. Finally, for any $u \in \operatorname{dom} \mathbf{P}_\lambda$, we use that $\mathbf{P}_\lambda$ is self-adjoint and obtain

$$\mathbf{P}_\lambda u = \sum_{j=1}^{\infty} (\mathbf{P}_\lambda u, v_j) v_j = \sum_{j=1}^{\infty} (u, (\mathbf{P} - \bar{\lambda} I) v_j) v_j = \sum_{j=1}^{\infty} (\lambda_j - \lambda)(u, v_j) v_j,$$

which implies that $A_\lambda \mathbf{P}_\lambda = Id$ and thus that $\mathbf{A}_\lambda$ is an (everywhere defined, bounded) inverse of $\mathbf{P} - \lambda I$, i.e. λ is not in the spectrum. □

4.5.4 Operators of Sublaplace type on CR manifolds

We want to make the Rockland condition for hypoellipticity more computable in the case of strictly pseudoconvex CR-manifolds (M^{2m+1}, H, J, η) and operators of Sublaplace type. This section mostly follows [Pon08, section 3.4].

Definition. An operator of *Sublaplace type* is a differential operator of (Heisenberg) order 2 with model operator

$$P^a = -\sum_{j=1}^{2m} (X_j^a)^2 - i\mu(a) X_0^a, \tag{4.34}$$

at every point $a \in M$, where $X_0, X_1, \dots, X_{2m}$ is an H-frame and μ is a (local) smooth section of $\mathrm{End}(E)$.

Let (M, H, J, η) be a strictly pseudoconvex CR manifold and $(X_0 = \xi, X_1, \dots, X_{2m})$ a local ON basis such that $(X_1, \dots, X_{2m})$ are an adapted orthonormal frame for H, i.e. $JX_j = X_{j+m}$ for $j = 1, \dots, m$. In this frame, the matrix $L = (L_{jk})$ of the Levi form, i.e.

$$\mathcal{L}_a(X_j, X_k) = -d\eta(X_j, X_k)\xi = L_{jk}\xi,$$

has the form

$$L = 2 \begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where I_m is the identity matrix of dimension m. As usual, we denote the model vector fields

$$X_j^a(g) = \frac{d}{dt} g \cdot exp(tX_j(a))|_{t=0}.$$

The commutator relations of $\mathfrak{t}_H\mathfrak{m}_a$ are given by

$$[X_j^a, X_k^a] = \mathcal{L}_a(X_j, X_k) = (-2\delta_{j,k-m} + 2\delta_{j-m,k}) X_0^a$$

for $j,k = 1,\dots,2m$ and all others are zero. Recall that the tangent group $T_H M_a$ is isomorphic to the Heisenberg group $\mathcal{H}^m$ for (and only for) contact manifolds. In particular, this is the case for strictly pseudoconvex CR manifolds. Using the Heisenberg group structure (1.1) and the basis (1.3) of its Lie algebra $\mathfrak{h}^m$, we have the commutator relations

$$[X_j, Y_k] = \delta_{jk} Z$$

and all others are zero. Thus, an isomorphism $F\colon \mathfrak{t}_H\mathfrak{m}_a \to \mathfrak{h}^m$ is given by

$$F(X_j^a) = -\sqrt{2}X_j, \quad F(X_{j+m}^a) = \sqrt{2}Y_j, \quad \text{and} \quad F(X_0) = Z \quad (j = 1, ..., m)$$

The Heisenberg group has two families of representations that we introduced in (3.32 ff.). First, we have a one-parameter family $\pi^\lambda\colon \mathcal{H}^m \to U(L^2(\mathbb{R}^m, \mathbb{C}))$ that descends to an induced representation π_*^λ of $\mathfrak{h}_m$ on $L^2(\mathbb{R}^m)$ given by

$$\begin{aligned}
&\pi_*^\lambda(Z)u(x) = 2\pi i\lambda u(x),\\
&\pi_*^\lambda(X_j)u(x) = -\frac{\partial u}{\partial x_j}(x), \qquad \pi_*^\lambda(Y_j)u(x) = -2\pi i\lambda x_j u(x) \quad (j = 1,\dots,m).
\end{aligned}$$

For this family, the space of smooth vectors $C^\infty(\pi^\lambda)$ is the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m)$. Second, we have the family of one-dimensional representation $\pi^\beta\colon \mathcal{H}^m \to U(\mathbb{C})$ parametrised by $\beta = (\beta_1,\dots,\beta_{2m})$ that descend to representations π_*^β of $\mathfrak{h}_m$ on $\mathbb{C}$ given by

$$\pi_*^\beta(Z)z = 0, \quad \pi_*^\beta(X_j)z = 2\pi i\beta_{2j-1}z, \quad \pi_*^\beta(Y_j)z = 2\pi i\beta_{2j}z \quad (j = 1,\dots,m).$$

Using the isomorphism F, these induce the following representations of $\mathfrak{t}_H\mathfrak{m}_a$:

$$\begin{aligned}
&\hat\pi_*^\beta(X_0^a) = 0 && \hat\pi_*^\lambda(X_0^a) = 2\pi i\lambda &&\\
&\hat\pi^\beta(X_j^a) = -2\sqrt{2}\pi i\beta_{2j-1} && \hat\pi_*^\lambda(X_j^a) = \sqrt{2}\frac{\partial}{\partial x_j} && (j = 1,\dots,m)\\
&\hat\pi_*^\beta(X_{j+m}^a) = 2\sqrt{2}\pi i\beta_{2j} && \hat\pi_*^\lambda(X_{j+m}^a) = -2\sqrt{2}\pi i\lambda x_j && (j = 1,\dots,m).
\end{aligned}$$

Thus, we obtain for an operator P of Sublaplace type

$$\hat\pi_*^\beta(P^a) = -8\pi^2\|\beta\|_2^2,$$

which is injective as long as $\beta \neq 0$. Considering the other family of representations, we have

$$\begin{aligned}
\hat\pi_*^\lambda(P^a) &= -\sum_{j=1}^m \left(2\frac{\partial^2}{\partial x_j^2} + 2(2\pi i\lambda)^2 x_j^2\right) - i\mu(a)2\pi i\lambda\\
&= -2\sum_{j=1}^m \left(\frac{\partial^2}{\partial x_j^2} - (2\pi\lambda)^2 x_j^2\right) + \mu(a)2\pi\lambda.
\end{aligned}$$

After a change of coordinates $y = \sqrt{2\pi|\lambda|}x$ (recall $\lambda \neq 0$) on $\mathbb{R}^m$, we can write this operator as

$$\begin{aligned}\hat{\pi}_*^\lambda(P^a) &= -2\sum_{j=1}^m \left(2\pi|\lambda|\frac{\partial^2}{\partial y_j^2} - (2\pi\lambda)^2\frac{y_j^2}{2\pi|\lambda|}\right) + \mu(a)2\pi\lambda \\ &= (2\pi\lambda)\left[-2\,\mathrm{sgn}(\lambda)\sum_{j=1}^m\left(\frac{\partial^2}{\partial y_j^2} - y_j^2\right) + \mu(a)\right],\end{aligned}$$

where sgn is the sign function, i.e. $\mathrm{sgn}(\lambda) = \frac{\lambda}{|\lambda|}$. The constant factor $2\pi\lambda \neq 0$ has no importance for the injectivity, so we can forget it. The eigenvalues of the harmonic oscillator $\mathrm{Os} = \sum_{j=1}^m \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right)$ are given by the set

$$\Lambda = \left\{ m + \sum_{j=1}^m \nu_j \;\middle|\; \nu_j \in \mathbb{N}_0 \right\} \tag{4.35}$$

and the corresponding eigenfunctions are rapidly decreasing. The operator $\pi_*^\lambda(P^a)$ is injective if and only if $2\,\mathrm{sgn}(\lambda)\,\mathrm{Os}\,u \neq \mu(a)u$ for any $u \in \mathcal{S}(\mathbb{R}^m)^{\oplus r}$. For scalar operators, this means that $\mu(a) \notin \pm 2\Lambda$. For bundle-valued operators, we use a suitable basis of $E_a \simeq \mathbb{C}^r$ to bring $\mu(a)$ into the form (over $\mathbb{C}$):

$$\mu(a) = \begin{pmatrix} \mu_1(a) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mu_r(a) \end{pmatrix}$$

where $\mu_1(a), \ldots, \mu_r(a)$ are the eigenvalues of $\mu(a)$ counted with multiplicities. If some of the eigenvalues of $\mu(a)$ are in $\pm\Lambda$, we order them such that

$$\mu_1(a), \ldots, \mu_s(a) \in \Lambda \quad \text{and} \quad \mu_{s+1}(a), \ldots, \mu_r(a) \notin \Lambda. \tag{4.36}$$

This is possible by using the Jordan normal form of $\mu(a)$ and suitably arranging the blocks.

As Os is diagonal, we obtain that $\pi_*^\lambda(P^a)$ is injective if and only if $\mathrm{spec}(\mu(a)) \cap -2\,\mathrm{sgn}(\lambda)\Lambda = \emptyset$. This can be seen as follows: Consider the equation

$$\frac{1}{2\pi\lambda}\pi_*^\lambda(P^a)u = \begin{pmatrix} -2\,\mathrm{sgn}(\lambda)\,\mathrm{Os} + \mu_1(a) & * & * \\ 0 & \ddots & * \\ 0 & 0 & -2\,\mathrm{sgn}(\lambda)\,\mathrm{Os} + \mu_r(a) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} = 0.$$

If $\mathrm{spec}(\mu(a)) \cap 2\,\mathrm{sgn}(\lambda)\Lambda = \emptyset$, then from the last line it follows that $u_r = 0$ and inductively all other components must be zero as well. If, on the other hand the, intersection is not empty, we choose an eigenfunction of $2\,\mathrm{sgn}(\lambda)\,\mathrm{Os}$ associated with the eigenvalue $\mu_1(a)$ (recall the order (4.36) of the eigenvalues). Then, $(u_1, 0, \dots, 0)^T \neq 0$ is in the kernel of $-2\operatorname{sgn}(\lambda)\operatorname{Os} + \mu(a)$.

Finally, going to the transpose $(P^a)^t = (P^t)^a$ only changes the sign of $\mu(a)$ which has no effect on the condition. Altogether, we have

Proposition 4.5.15 ([Pon08, Prop 3.4.1, 3.4.3, 3.4.5])**.** *For an operator P of Sublaplace type, given in the form (4.34), the Rockland condition for P^a and $(P^t)^a$ at each point is equivalent to*

$$\operatorname{spec}\mu(a) \cap \left\{ \pm 2\left(m + \sum_{j=1}^{m} \nu_j\right) \,\middle|\, \nu_j \in \mathbb{N}_0 \right\} = \emptyset. \tag{4.37}$$

Thus, P is Heisenberg-elliptic (and thus admits a parametrix) if and only if this condition holds.

Remark. This proposition may be proven without the general theory. See [BG88, Thm 18.4] for the scalar case and [Pon08, Prop 3.4.5].

We now turn our attention to the horizontal Dirac operators over spin contact metric manifolds we want to study. In a local frame $(X_0 = \xi, X_1, \dots X_{2m})$ as above, it is locally given by

$$D_H^{\nabla} = \sum_{j=1}^{2m} X_j \cdot \nabla_{X_j},$$

where the multiplication is Clifford multiplication. Going to a trivialisation of $\mathbb{S}$, we have that ∇_{X_j} is given by X_j+ endomorphism terms. Thus, the model operator is given by

$$(D_H^{\nabla})^a = \sum_{j=1}^{2m} cl(X_j^a) X_j^a,$$

where cl denotes Clifford multiplication. Note that this is independent of the covariant derivative ∇ chosen. By definition of π_* on operators, we have $\pi_*(P^2) = (\pi_*(P))^2$. Obviously, a linear operator is injective if and only if its square is injective, so instead of D_H^{∇}, we can consider $(D_H^{\nabla})^2$ and apply the above theory for operators of Sublaplace type. Using the product structure of $U(\mathfrak{t}_H\mathfrak{m}_a)$, we obtain (where we omit the superscript a in X_j^a)

$$\begin{aligned}((D_H^\nabla)^2)^a = ((D_H^\nabla)^a)^2 &= \sum_{j=1}^{2m} cl(X_j)cl(X_j)X_j^2 + \sum_{1\le j<k\le 2m} cl(X_jX_k)(X_jX_k - X_kX_j)\\ &= -\sum_{j=1}^{2m} X_j^2 - \sum_{1\le j<k\le 2m} cl(X_jX_k)d\eta(X_j,X_k)\xi\\ &= -\sum_{j=1}^{2m} X_j^2 - cl(d\eta)\xi. \end{aligned} \tag{4.38}$$

Remark. On a side note, the presence of the ξ-derivative in the model operator shows that it will be impossible to obtain a Weitzenböck type formula for D_H^∇ in the usual sense. From the local formula for the horizontal connection Laplacian (3.7), one immediately deduces that its model operator is $(\Delta_H^\nabla)^a = \sum_{j=1}^{2m} -X_j^2$. The presence of the ξ-derivative in our Weitzenböck formula (3.12) is therefore not due to a bad choice of the connection defining the connection Laplacian. Also, the first-order derivative in the transversal direction must be "hidden" in the CR connection Laplacians in (3.16).

The model operator $((D_H^\nabla)^a)^2$ is thus of the form (4.34) with $\mu(a) = -id\eta$. The Clifford multiplication with $id\eta$ has eigenvalues $-2m, -2m+2, \dots, 2m-2, 2m$ and thus, we see that $(D_H^\eta)^2$ and $(D_H^\nabla)^2$ are not Heisenberg-elliptic. This was already noted in [Has14, section 6.3]. In the case of the Tanaka-Webster operator D_H^η, i.e. the horizontal Dirac operator associated with the Tanaka-Webster connection, we can, however, refine this result.

The spinor bundle $\mathbb{S}$ splits under the Clifford action of $d\eta$ into eigenspaces (cf Proposition 2.3.1)

$$\mathbb{S} = \bigoplus_{k=0}^{m} \mathbb{S}_{m-2k}, \quad \mathbb{S}_{m-2k} = \{\varphi \in \mathbb{S} \mid \, d\eta \cdot \phi = (m-2k)2i\varphi\}. \tag{4.39}$$

As we have shown in section 3.2, the square of the Tanaka-Webster operator commutes with the Clifford action of $d\eta$, thus stabilising the eigenspaces. We can thus consider the operators

$$(D_H^\eta)^2|_{\Gamma(\mathbb{S}_{m-2k})} \colon \Gamma(\mathbb{S}_{m-2k}) \to \Gamma(\mathbb{S}_{m-2k}).$$

On these bundles, there is only one eigenvalue, $(m-2k)2i$, of $d\eta$ and thus, condition (4.37) is satisfied if $k \neq 0, m$. We obtain the following result.

Proposition 4.5.16. *Let (M,H,J,η) be a spin strictly pseudoconvex CR manifold and D_H^η the horizontal Dirac operator induced by its Tanaka-Webster connection.*

Then, the restrictions of $(D_H^\eta)^2$ *to* $\Gamma(\mathbb{S}_{(m-2k)})$ *are hypoelliptic for* $k = 1, \dots, m-1$. *If* M *is closed, they extend to Fredholm operators* $(D_H^\eta)^2_s \colon L^2_s(\mathbb{S}_{(m-2k)}) \to L^2_{s-k}(\mathbb{S}_{(m-2k)})$. *In these cases, the spectrum consists only of real, nonnegative eigenvalues that tend to infinity. The corresponding eigenspaces are finite-dimensional, consist of smooth sections and span* $L^2(\mathbb{S}_{m-2k})$.

Proof. The hypoellipticity has been proven above. The spectral properties then follow from the corresponding properties of Heisenberg-elliptic operators. That the eigenvalues are nonnegative is a consequence of the fact that $(D_H^\eta)^2$ is the square of an operator. □

Remark. Note that this is an analogous result as for the Kohn-Laplacian on strictly pseudoconvex CR manifold as stated in [BG88, section 21]. In fact, it may be shown that if we replace the *Spin* structure on M with a *Spin*^{c} structure, then for a suitable choice of auxiliary connection, the square of the Tanaka-Webster operator and the Kohn Laplacian agree. On the other hand, the symbols of the *Spin* and *Spin*^{c} Dirac operators are the same.

That D_H^η is not hypoelliptic on the two extremal bundles $\mathbb{S}_{\pm m}$ does not mean that we need to give up hope for a “nice” spectrum totally. Remembering the close relationship of $(D_H^\eta)^2$ and the Kohn Laplacian, in the next section we will try to adapt the result on the existence of partial inverses and the consequences for the spectrum of the Kohn Laplacian.

4.5.5 The Tanaka-Webster operator on the extremal bundles

In this section, we will discuss the spectrum for the (square of the) Tanaka-Webster operator on the “extremal” bundles $\mathbb{S}_{\pm m}$. We restrict ourselves to the case $m > 1$. In this case, $(D_H^\eta)^2$ may not have a parametrix on the extremal bundles, but we are still able to construct a partial inverse from the parametrices on the other parts of the spinor bundle, where $(D_H^\eta)^2$ is Heisenberg-elliptic. This technique is not available in dimension 3 (i.e. $m = 1$) because the spinor bundle consists exclusively of the extremal parts. The discussion in this section is adapted from analogous arguments for the Kohn Laplacian in [BG88, section 24].

Let us recall (cf Sections 3.1 and 3.2) that the horizontal Dirac operator splits as

$$D_H^\eta = D_+^\eta + D_-^\eta, \quad D_\pm^\eta \colon \Gamma(\mathbb{S}_{m-2k}) \to \Gamma(\mathbb{S}_{m-2(k\mp1)}),$$

where we set $\mathbb{S}_{-m-2} = \mathbb{S}_{m+2} = 0$. We also remind the reader that these operators have the following properties:

$$(D_\pm^\eta)^2 = 0, \quad (D_H^\eta)^2 = D_+^\eta D_-^\eta + D_-^\eta D_+^\eta, \quad \text{and} \quad (D_\pm^\eta)^* = D_\mp^\eta. \tag{4.40}$$

The idea for constructing partial inverses for D_H^η on $\mathbb{S}_{-m}$ and $\mathbb{S}_m$ is to go to another eigenbundle where D_H^η is H-elliptic via $D_\pm^\eta$ and use the partial inverses that we know exist there by Prop 4.5.11. In preparation, we collect some identities for $D_\pm^\eta$, the projections and partial inverses. These identities are the D_H^η-analogues of the identities for the Kohn Laplacian $\Box_b$ in [BG88, Lemma 24.9].

Lemma 4.5.17. *Let (M, H, J, η) be a closed spin strictly pseudoconvex CR manifold and D_H^η the horizontal Dirac operator induced by its Tanaka-Webster connection. We denote $D^k = (D_H^\eta)^2|_{\Gamma(\mathbb{S}_{m-2k})}$, $D_\pm^k$ analogously and write N^k, Π_1^k, Π_2^k $(k = 1, \ldots, m-1)$ for the H-pseudodifferential operators that provide the partial inverse and projections of D^k. Then, we have the following identities in Ψ_H^*:*

$$\Pi_1^k = \Pi_2^k, \tag{4.41}$$

$$D_+^k \Pi_1^k = D_+^k, \qquad D_-^k \Pi_1^k = D_-^k, \tag{4.42}$$

$$\Pi_1^k D_+^{k+1} = D_+^{k+1}, \qquad \Pi_1^k D_-^{k-1} = D_-^{k-1}, \tag{4.43}$$

$$\begin{aligned} \Pi_1^k D_-^{k-1} D_+^k &= D_-^{k-1} D_+^k & \Pi_1^k D_+^{k+1} D_-^k &= D_+^{k+1} D_-^k \\ &= D_-^{k-1} D_+^k \Pi_1^k, & &= D_+^{k+1} D_-^k \Pi_1^k, \end{aligned} \tag{4.44}$$

$$N^k D_-^{k-1} D_+^k = D_-^{k-1} D_+^k N^k, \qquad N^k D_+^{k+1} D_-^k = D_+^{k+1} D_-^k N^k. \tag{4.45}$$

Proof. By Propositions 4.5.10 and 4.5.11, $\mathbf{D}^k$ is self-adjoint. This implies the first identity. Now let $\varphi \in \ker \mathbf{D}^k$. By the regularity properties of D^k, φ is smooth and we obtain

$$0 = ((D_H^\eta)^2 \varphi, \varphi) = (D_+^k \varphi, D_+^k \varphi) + (D_-^k \varphi, D_-^k \varphi).$$

As both terms on the right hand side are nonnegative, this implies

$$\ker(\mathbf{D}^k) = \ker(D_+^k) \cap \ker(D_-^k) \cap \Gamma(\mathbb{S}_{m-2k}),$$

where $D_\pm^k$ are to be understood as operators on distributions. As $\ker \Pi_1^k = \ker \mathbf{D}^k$, equations (4.42) follow. As an orthogonal projection, Π_j is self-adjoint and thus, by taking adjoints equations (4.43) follow on smooth spinors and thus as identities in Ψ_H^*. The equations (4.44) are an immediate consequence. Using the identities we have already established, we find that

$$\begin{aligned} N^k D_+^{k+1} D_-^k &= N^k D_+^{k+1} D_-^k \Pi_1^k \\ &= N^k D_+^{k+1} D_-^k (D_+^{k+1} D_-^k + D_-^{k-1} D_+^k) N^k \\ &\overset{(4.40)}{=} N^k (D_+^{k+1} D_-^k + D_-^{k-1} D_+^k) D_+^{k+1} D_-^k N^k \\ &= \Pi_1^k D_+^{k+1} D_-^k N^k = D_+^{k+1} D_-^k N^k. \end{aligned}$$

This proves the first equation in (4.45) and the second one is proved analogously. $\square$

We are now ready to prove that $(D_H^\eta)^2$ (i.e. its L^2-realisation) on the extremal bundles $\Gamma(\mathbb{S}_{-m})$ and $\Gamma(\mathbb{S}_m)$ admits a partial inverse. The following proposition is the D_H^η-analogue of [BG88, Theorem 24.20] for $\Box_b$.

Proposition 4.5.18. *Let (M, H, J, η) be a closed spin strictly pseudoconvex CR manifold of dimension $2m+1 \geq 5$ and D_H^η the horizontal Dirac operator induced by its Tanaka-Webster connection. Denote D^k the restriction of $(D_H^\eta)^2$ to $\Gamma(\mathbb{S}_{m-2k})$ and N^k, Π_j^k the associated partial inverse and projections. Then, the L^2-realisation $\mathbf{D}^m$ has partial inverse induced by $D_-^\eta(N^{m-1})^2 D_-^\eta \in \Psi^{-2}(\mathbb{S}_{-m})$ and the associated projections are equal and induced by $D_-^\eta N^{m-1} D_+^\eta \in \Psi^0(\mathbb{S}_{-m})$.*

The restriction to the other extremal bundle, $\mathbf{D}^0$, has partial inverse induced by $D_+^\eta(N^1)^2 D_-^\eta \in \Psi^{-2}(\mathbb{S}_m)$ and the associated projections are equal and induced by $D_+^\eta N^1 D_-^\eta \in \Psi^0(\mathbb{S}_m)$.

Proof. We prove that the operators defined above satisfy the assumptions of Proposition 4.5.10. We begin with the projections. We have for $k = 1, m-1$ that $(N^k D^k)^* = (\Pi_1)^* = \Pi_1 = \Pi_2$ and on the other hand, $(N^k D^k)^* = (D^k)^*(N^k)^* = D^k(N^k)^*$, i.e. $D^k(N^k)^* = \Pi_2$. Analogously, one obtains $(N^k)^* D^k = \Pi_1$ by taking the (formal) adjoints of $D^k N^k = \Pi_2$. As the partial inverse is unique, this implies $(N^k)^* = N^k$. Then, the (formal) self-adjointness of the proposed projections follows from this and the fact that $(D_\pm^\eta)^* = D_\mp^\eta$. Moreover, writing $D_\pm^k$ for the restrictions of $D_\pm^\eta$ to $\Gamma(\mathbb{S}_{(m-2k)})$, we have

$$
\begin{aligned}
(D_-^{m-1} N^{m-1} D_+^m)^2 &= D_-^{m-1} N^{m-1} (D_+^m D_-^{m-1}) N^{m-1} D_+^m \\
&\overset{(4.45)}{=} D_-^{m-1} N^{m-1} N^{m-1} D_+^m D_-^{m-1} D_+^m \\
&\overset{(4.40)}{=} D_-^{m-1} N^{m-1} N^{m-1} (D_-^{m-2} D_+^{m-1} + D_+^m D_-^{m-1}) D_+^m \\
&= D_-^{m-1} N^{m-1} \Pi_1^{m-1} D_+^m \\
&\overset{(4.43)}{=} D_-^{m-1} N^{m-1} D_+^m.
\end{aligned}
$$

An analogous argument holds for $D_+^1 N^1 D_-^0$. Thus, (4.33) is satisfied. Next, we have

$$
\begin{aligned}
D^m (D_-^{m-1} N^{m-1} N^{m-1} D_+^m) &= D_-^{m-1} D_+^m D_-^{m-1} N^{m-1} N^{m-1} D_+^m \\
&\overset{(4.40)}{=} D_-^{m-1} (D_+^m D_-^{m-1} + D_-^{m-2} D_+^{m-1}) N^{m-1} N^{m-1} D_+^m \\
&= D_-^{m-1} \Pi_1^{m-1} N^{m-1} D_+^m \\
&\overset{(4.42)}{=} D_-^{m-1} N^{m-1} D_+^m
\end{aligned}
$$

and

$$\begin{aligned}
(D_-^{m-1}N^{m-1}N^{m-1}D_+^m)D^m &= D_-^{m-1}N^{m-1}N^{m-1}D_+^mD_-^{m-1}D_+^m \\
&\overset{(4.40)}{=} D_-^{m-1}N^{m-1}N^{m-1}(D_+^mD_-^{m-1}+D_-^{m-2}D_+^{m-1})D_+^m \\
&= D_-^{m-1}N^{m-1}\Pi_1^{m-1}D_+^m \\
&\overset{(4.43)}{=} D_-^{m-1}N^{m-1}D_+^m.
\end{aligned}$$

Again, the arguments for D^0 are completely analogous and (4.31) is satisfied. Finally,

$$\begin{aligned}
(D_-^{m-1}N^{m-1}D_+^m)D^m &= D_-^{m-1}N^{m-1}D_+^mD_-^{m-1}D_+^m \\
&\overset{(4.40)}{=} D_-^{m-1}N^{m-1}(D_+^mD_-^{m-1}+D_-^{m-2}D_+^{m-1})D_+^m \\
&= D_-^{m-1}\Pi_1^{m-1}D_+^m \qquad (4.46)\\
&= D_-^{m-1}(D_-^{m-2}D_+^{m-1}+D_+^mD_-^{m-1})N^{m-1}D_+^m \\
&\overset{(4.40)}{=} D_-^{m-1}D_+^mD_-^{m-1}N^{m-1}D_+^m \\
&= D^m(D_-^{m-1}N^{m-1}D_+^m).
\end{aligned}$$

By (4.43), (4.46) is equal to D^m and thus (4.32) is satisfied for D^m. As before, the arguments for D^0 are completely analogous. □

The existence of a partial inverse on the extremal bundles allows us to conclude that $(D_H^\eta)^2$ has pure point spectrum.

Theorem 4.5.19. *Let (M,H,J,η) be a closed spin strictly pseudoconvex CR manifold of dimension $2m+1\geq 5$ and D_H^η the horizontal Dirac operator induced by its Tanaka-Webster connection. Then, the restrictions of $(D_H^\eta)^2$ to the extremal bundles $\Gamma(\mathbb{S}_{-m})$ and $\Gamma(\mathbb{S}_m)$ have pure point spectrum. The eigenvalues are real, nonnegative, tend to infinity and the eigenspaces associated to the nonzero eigenvalues are finite-dimensional and consist of smooth functions. Moreover, there exists an L^2-basis of eigenfunctions (φ_j).*

Proof. Throughout this proof, let D be the restriction of $(D_H^\eta)^2$ to one of the extremal bundles and $\mathbf{D}$ its L^2-realisation. It follows from propositions 4.5.10 and 4.5.18 that $\mathbf{D}$ is self-adjoint and $\operatorname{ran}(\mathbf{D})=\ker\mathbf{D}^\perp$. On $\ker\mathbf{D}^\perp$, the partial inverse $\mathbf{A}$ is an inverse for $\mathbf{D}$. As $\mathbf{A}$ is a pseudodifferential operator of negative order, it is compact and self-adjointness carries over from $\mathbf{D}$ to its inverse $\mathbf{A}$. Thus, $\mathbf{A}$ has pure point spectrum $(\mu_j)_j$, $|\mu_j|\to 0$ as $j\to\infty$, and an associated orthonormal system $(\phi_j)_j$ spanning $\ker\mathbf{D}^\perp$ such that $\mathbf{A}\phi_j=\mu_j\phi_j$ (the μ_j are counted with multiplicities). Then, the ϕ_j are eigenspinors of $\mathbf{D}$ associated to the eigenvalues $\frac{1}{\mu_j}$. As $|\mu_j|\to 0$, any eigenvalue appears only a finite number of times and thus, the eigenspace associated with any eigenvalue is finite-dimensional. Let φ be an eigen-section. Then, as $A \in \Psi_H^{-2}(\mathbb{S}_{\pm m})$, $A\varphi \in L_1^2(\mathbb{S}_{\pm m})$. As $A\varphi = \mu_j\varphi$, φ must itself be in L_1^2. By induction, one sees that $\varphi \in L_k^2(\mathbb{S}_{\pm m})$ for any $k \in \mathbb{N}$ and thus, φ is smooth. As D is self-adjoint, the eigenvalues are real and as it is the square of D_H^η, the eigenvalues are nonnegative.

Now, combining (φ_j) with an L^2-basis of $\ker \mathbf{D}$, we obtain an L^2-basis (ψ_j) of $L^2(\mathbb{S}_{\pm m})$ such that $\mathbf{D}\psi_j = \lambda_j\psi_j$, where $\lambda_j = 0$ possibly appears infinitely often and every other eigenvalue only finitely often. We are left to show that any other value λ cannot be in the spectrum. This follows from Proposition 3.5.2. $\square$

Corollary 4.5.20. *Let (M, H, J, η) be a closed spin strictly pseudocovex CR manifold of dimension $2m+1 \geq 5$ and D_H^η the horizontal Dirac operator induced by its Tanaka-Webster connection. Then, $(D_H^\eta)^2$ has pure point spectrum, the eigenvalues are real, nonnegative and tend to infinity. The eigenspaces associated with the nonzero eigenvalues are finite dimensional and consist of smooth sections of the spinor bundle. The same holds for* $\ker((D_H^\eta)^2) \cap L^2(\mathbb{S}_{m-2k}) = \ker((D_H^\eta)^2) \cap \Gamma(\mathbb{S}_{m-2k})$ *for $k \neq 0, m$.*

Appendix: Some functional analysis

In this appendix we collect some facts from functional analysis (Fourier transform, distributions, Fredholm operators and Sobolev spaces) that are used in the main text.

A.1 Test functions, distributions and the Fourier transform

Most results of this section are well-established and can be found in any textbook on functional analysis. Unless otherwise noted, the presentation here follows [Ler]. The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, or space of rapidly decreasing functions, is the space of all smooth functions $f \in C^\infty(\mathbb{R}^n)$ such that for all multi-indices α, β

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (\tfrac{\partial}{\partial x})^\beta f(x)| < \infty.$$

Obviously, the space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n)$ is contained in $\mathcal{S}(\mathbb{R}^n)$. The space of rapidly decreasing functions is equipped with the countable family of seminorms

$$p_k(f) = \sup_{|\alpha|,|\beta|\leq k,\ x\in\mathbb{R}^n} |x^\alpha (\tfrac{\partial}{\partial x})^\beta f(x)|.$$

These seminorms make $\mathcal{S}(\mathbb{R}^n)$ into a Fréchet space, i.e. its topology can be defined via a metric and the resulting metric space is complete. If we have a countable family of seminorms (p_k) (as for $\mathcal{S}(\mathbb{R}^n)$), a metric is given by

$$d(f,g) = \sum_{k=1}^\infty 2^{-k} \frac{p_k(f-g)}{1+p_k(f-g)}.$$

On the Schwartz space one can define the *Fourier transform* $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ given for $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx.$$

The Fourier transform of a function f is often denoted $\hat{f}$.

Lemma A.1.1. *The Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous linear isomorphism with inverse given by*

$$(\mathcal{F}^{-1}f)(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi) d\xi.$$

An important property of the Fourier transform that is used a lot in the analysis of operators is that under the Fourier transform, deriving corresponds to multiplying.

Lemma A.1.2. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then,*

$$\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(x) = (2i\pi) \cdot \xi_j \cdot (\mathcal{F}f)(\xi)$$

Distributions, sometimes called *generalised functions*, are elements of the topological dual of a function space. The following example may motivate the following definition of distributions: Consider the smooth functions $f_n\colon \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} ne^{-\frac{1}{1-(nx)^2}} & \text{for } |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, the limit of these functions does not exist, as it would be a function with value ∞ at zero that vanishes everywhere else. However, if we interpret the functions f_n as functionals on on the space of smooth function $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ via

$$g \mapsto \int_{\mathbb{R}} f_n(x)g(x)dx,$$

the limit does exist and is given by a constant multiple of the Dirac delta $\delta_0\colon \mathcal{E}(\mathbb{R}^n) \to \mathbb{R}$, $\delta_0(g) = g(0)$ (note that $\int_{-1/n}^{1/n} f_n(x)dx$ is independent of n).

The topological dual V' of a topological vector space V is the set of all continuous linear maps $L\colon V \to \mathbb{R}$. We begin by discussing the topologies of the function spaces $\mathcal{E}(U) = C^\infty(U)$ and $\mathcal{D}(U) = C_c^\infty(\mathbb{R}^n)$. The space of smooth functions $\mathcal{E}(U)$ is a Fréchet space with topology defined by the seminorms

$$p_{j,k}(f) = \sup_{|\alpha|\le j, \|x\|_2 \le k} |(\tfrac{\partial}{\partial x})^\alpha f(x)|.$$

Unfortunately, there is no countable family of seminorms defining “the” topology of $\mathcal{D}(U)$. Instead, we proceed as follows: For any compact $K \subset U$, the space of smooth functions with support in K, $\mathcal{E}_K(U) \subset \mathcal{E}(U)$, can be equipped with the induced topology. We then have that

$$\mathcal{D}(U) = \bigcup_{K\subset U \text{ compact}} \mathcal{E}_K(U)$$

and we can equip $\mathcal{D}(U)$ with the inductive limit topology. The inductive limit topology (compare [Con84, Section IV.5]) consists of all sets V such that for every $x \in V$, there is a convex set $W \subset X$ such that $W \cap \mathcal{E}_K(U)$ is open in $\mathcal{E}_K(U)$ for all K, $\lambda y \in W$ for all $y \in W$ and $\lambda \in (-1, 1)$ and $x + W \subset V$.

In particular, by a *distribution* one usually means an element of the topological dual of $\mathcal{D}(U)$. The continuity of a functional $L\colon \mathcal{D}(U) \to \mathbb{R}$ can be characterised as follows: L is continuous if and only if for every compact set $K \subset U$, there exist constants $C_K > 0$ and $n_k \in \mathbb{N}$ such that for every smooth function f with support in K,

$$|L(f)| \leq C_k \sup_{|\alpha| \leq n_k, x \in K} |(\tfrac{\partial}{\partial x})^\alpha f(x)|.$$

Any function $f \in \mathcal{D}(U)$ defines a distribution $L_f \in \mathcal{D}'(U)$ via

$$L_f(g) = \int_U f(x) g(x) dx.$$

In fact, f need not have compact support, any function $f \in C^\infty(U)$ or even $f \in L^1_{loc}(U)$ will do.

The space of *distributions with compact support* $\mathcal{E}'(U)$ is the topological dual of $\mathcal{E}(U)$. There is an inclusion $\mathcal{D}(U) \hookrightarrow \mathcal{E}'(U)$. In general, a function $f \in \mathcal{E}(U)$ does not define a distribution with compact support.

Now, let $P\colon \mathcal{D}(U) \to \mathcal{E}(U)$ be a continuous operator (for example, a pseudodifferential operator). Then, the *transpose operator* is the operator $P^t\colon \mathcal{E}'(U) \to \mathcal{D}'(U)$ defined by

$$(P^t L)(g) = L(Pg) \quad \text{for any } L \in \mathcal{E}'(U), g \in \mathcal{D}(U).$$

For $f \in \mathcal{D}(U)$, we obtain for any $g \in \mathcal{D}(U)$

$$(P^t L_f)(g) = L_f(Pg) = \int_U f(Pg)\, dx.$$

Thus, if $P^t f$ is again a smooth function (which is anything but guaranteed in general), it is uniquely (because $\mathcal{D}(U) \subset \mathcal{E}(U)$ is dense) defined by

$$\langle P^t f, g \rangle = \langle f, Pg \rangle \quad \text{for all } g \in \mathcal{D}(U),$$

where $\langle f, g \rangle = \int fg$. If P^t restricts to an operator $P^t\colon \mathcal{D}(U) \to \mathcal{E}(U)$, then P extends to an operator

$$P\colon \mathcal{E}'(U) \to \mathcal{D}'(U)$$

via $(PL)(f) = L(P^t F)$. By the arguments above, for $f \in \mathcal{D}(U)$, $(PL_f)(g) = L_{Pf}(g)$.

The space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^n)$ is the dual of the Schwartz functions. With an analogous definition as above, any Schwartz function is a tempered distribution and we obtain the following inclusions:

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

The Fourier transform can be extended to $\mathcal{S}'(\mathbb{R}^n)$ via the following definition: For any $L \in \mathcal{S}'(\mathbb{R}^n)$, define

$$(\mathcal{F}(L))(f) = L(\mathcal{F}(f)).$$

For $f \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}(L_f) = L_{\mathcal{F}(f)}$.

Lemma A.1.3. *The Fourier transform $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear isomorphism.*

Generalised sections on manifolds

We now move on to distributions on manifolds. The presentation of this material is based on [vdBC, Lecture 2]. We begin by describing the topology of $\Gamma(E)$ and $\Gamma_c(E)$. Consider a covering of M by open sets $(U_j)_{j\in J}$ equipped with trivialising charts $\kappa_j\colon E|_{U_j} \to V_j \times \mathbb{K}^r$, where $V_j \subset \mathbb{R}^n$ open. These trivialisations induce an injection

$$\Gamma(E) \hookrightarrow \coprod_{j\in J} \mathcal{E}(V_j)^r$$

and the topology is then induced on $\Gamma(E)$ by the product topology on the right-hand side. This construction is independent of the choices made and the resulting topology makes $\Gamma(E)$ into a Fréchet space. The topology of $\Gamma_c(E)$ is then obtained as before as the inductive limit topology of

$$\Gamma_c(E) = \bigcup_{K\subset M \text{ compact}} \Gamma_K(E).$$

Unlike on euclidean space, there is no canonical way to integrate functions on a general manifold and therefore, no canonical embedding of a function space in its topological dual. To circumvent this problem, one takes a tensor product with the density bundle before going to the dual. The $\mathbb{K}$-density bundle ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) is defined as associated vector bundle $|\Lambda(M)| = P_{GL(M)} \times_\rho \mathbb{K}$ via the representation $\rho\colon GL_n \to \mathbb{K}$, $\rho(A) = |\det(A)|$. It can be seen as a replacement for n-forms for the purpose of integration on non-oriented manifolds.

For a $\mathbb{K}$-vector bundle $E \to M$, the space of *generalised sections* $\mathcal{D}'(M, E)$ is then the topological dual of $\Gamma_c(E^* \otimes_{\mathbb{K}} |\Lambda(M)|) = \Gamma_c(\mathrm{Hom}(E, |\Lambda(M)|))$ and the space of *generalised sections with compact support* $\mathcal{E}'(E)$ is the topological dual of $\Gamma(E^* \otimes |\Lambda(M)|)$. For the trivial bundle $E = M \times \mathbb{K}$ (i.e. $\Gamma(E) = C^\infty(M)$), we write simply $\mathcal{D}'(M)$ and $\mathcal{E}'(M)$.

Now, any $s \in \Gamma_c(E)$ defines an Element of $L_s \in \mathcal{E}'(M, E)$ as follows: For $h \in \Gamma(\mathrm{Hom}(E, |\Lambda(M)|))$, $h(s)$ is a density with compact support on M and we can thus set

$$L_s(h) = \int_M h(s). \tag{A.1}$$

For a continuous operator $P\colon \Gamma_c(E) \to \Gamma(E)$, the *transpose operator* $P^t\colon \Gamma(E)' \to \Gamma_c(E)'$ is defined by $P^t(Ls) = L(Ps)$ for any $L \in \Gamma(E)'$ and $s \in \Gamma_c(E)$. Now, unfortunately in the manifold case $\Gamma_c(E)'$ is not $\mathcal{D}'(E)$. However, an easy calculation shows that for any $\mathbb{K}$-vector spaces U, V, W, with W of dimension one, we have

$$\mathrm{Hom}(U \otimes W, V \otimes W) = \mathrm{Hom}(U, V).$$

This extends to homomorphisms of vector bundles and we obtain

$$\mathrm{Hom}(E^* \otimes |\Lambda(M)|, |\Lambda(M)|) \simeq \mathrm{Hom}(E^*, \mathbb{K}) \simeq E.$$

Thus, $\mathcal{D}'(E^* \otimes |\Lambda(M)|) \simeq \Gamma_c(E)'$, $\mathcal{E}'(E^* \otimes |\Lambda(M)|) \simeq \Gamma(E)'$ and the transpose operator can be seen as an operator

$$P^t\colon \mathcal{E}'(E^* \otimes |\Lambda(M)|) \to \mathcal{D}'(E^* \otimes |\Lambda(M)|).$$

As there is an embedding $\Gamma_c(E^* \otimes |\Lambda(M)|) \hookrightarrow \mathcal{E}'(E^* \otimes |\Lambda(M)|)$ as before, P^t restricts to an operator

$$P^t\colon \Gamma_c(E^* \otimes |\Lambda(M)|) \to \mathcal{D}'(E^* \otimes |\Lambda(M)|).$$

If $P^t(\Gamma_c(E^* \otimes |\Lambda(M)|)) \subset \Gamma(E^* \otimes |\Lambda(M)|)$, then P extends to an operator

$$P\colon \mathcal{E}'(E) \to \mathcal{D}'(E)$$

by setting $(PL)(h) = L(P^t h)$ for any $L \in \mathcal{E}'(E)$ and $h \in \Gamma_c(E^* \otimes |\Lambda(M)|)$.

If the manifold M is oriented, any choice of a volume form ω trivialises $|\Lambda(M)|$ and thus $E^* \otimes |\Lambda(M)| \simeq E^*$ and the spaces of generalised sections are simply the topological duals of $\Gamma_{(c)}(E^*)$. In this case, the embedding (A.1) can be replaced as follows: For any $s \in \Gamma_c(E)$, define $L_s \in \Gamma(E)'$ via

$$L_s(e) = \int_M e(s)\omega \quad \text{for any } e \in \Gamma(E).$$

Note that this identification and this embedding are not canonical as they depend on the choice of a volume form. In the case of an oriented Riemannian manifold (M, g), there is the canonical choice of the Riemannian volume form dM^g and the problems vanish.

A.2 Fredholm operators

Definition. Let H_1, H_2 be Banach spaces and $A \in \mathcal{L}(H_1, H_2)$. Let $\mathrm{coker}(A) = H_2/\mathrm{ran}(A)$. The linear operator A is called *Fredholm* if both $\ker(A)$ and $\mathrm{coker}(A)$ are finite-dimensional. The *index* of a Fredholm operator is given by

$$\mathrm{index}(A) = \dim\ker(A) - \dim\mathrm{coker}(A).$$

The finite-dimensionality of the cokernel makes A an isomorphism on the complement of its kernel.

Lemma A.2.1 ([Shu01, Lem 8.1])**.** *Let* $A \in \mathcal{L}(H_1, H_2)$ *and* $\dim \operatorname{coker}(A) < +\infty$*. Then,* $\operatorname{ran}(A) \subset H_2$ *is closed.*

Corollary A.2.2 ([Shu01, Cor 8.1 and comments thereafter])**.** *Let* $A \in \mathcal{L}(H_1, H_2)$ *and* $\dim \operatorname{coker}(A) < +\infty$*. Let* $L \subset H_1$ *be a closed subspace such that* $H_1 = L \oplus \ker(A)$*, then* $A|_L \colon L \to \operatorname{ran}(A)$ *is a topological isomorphism.*

In particular, if $A \colon H_1 \to H_2$ *is Fredholm, such a space* L *always exists and thus* $A|_L \colon L \to \operatorname{ran}(A)$ *is a topological isomorphism.*

If the cokernel of A is finite-dimensional, it is given by the kernel of the transpose operator.

Lemma A.2.3 ([Shu01, Cor 8.2])**.** *Let* $A \in \mathcal{L}(H_1, H_2)$ *and* $\dim \operatorname{coker}(A) < +\infty$*. Then,*

$$\dim \operatorname{coker}(A) = \dim \ker(A^t),$$

where $A^t \colon H_2' \to H_1'$ *is the transpose operator. If* $H = H_1 = H^2$ *is a Hilbert space, the same holds for the adjoint operator* A^**.*

The following criterion will be helpful for showing that a given operator is Fredholm.

Proposition A.2.4 ([Shu01, prop 8.2])**.** *Let* $A \in \mathcal{L}(H_1, H_2)$ *and let there exist* B_1, B_2 *and compact operators* S_1, S_2 *such that*

$$B_1 A = I + S_1 \quad \text{and} \quad A B_2 = I + S_2.$$

Then A *is Fredholm.*

A.3 Sobolev spaces

The spaces of differentiable functions $C^k(U)$ with the usual C^k-norms are not complete. Taking their completion, one obtains the Sobolev spaces or spaces of "weakly differentiable functions". We introduce these spaces for sections of vector bundles and discuss their main properties.

There are weighted Sobolev spaces adapted to the Heisenberg structure, compare [Pon08, section 5.5]. We will instead stick to the standard Sobolev spaces as they will suffice for our purposes and follow the definitions as they are given in [LM89, section III.2].

Let E be a Hermitian vector bundle over a compact Riemannian manifold (M, g) with a covariant derivative ∇^E. Let M be equipped with its Levi-Civita connection

∇^g. Then, for $u \in \Gamma(E)$, $\nabla^E u \in \Gamma(T^*M \otimes E)$ and using the tensor product derivative of ∇^g and ∇^E, we obtain a covariant derivative on $\Gamma(T^*M \otimes E)$. Iterating and composing, we obtain

$$\nabla^{k,E} \colon \Gamma(E) \longrightarrow \Gamma(\underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ times}} \otimes E).$$

Then, we can define the *k-th Sobolev norm*

$$\|u\|_k = \|u\|_{L^2} + \sum_{j=1}^{k} \|\nabla^{k,E} u\|_{L^2}.$$

The completion of $\Gamma(E)$ with respect to this norm is called the *k-th Sobolev space* and denoted $L^2_k(E)$.

One can extend the definition of Sobolev spaces to real orders s. We sketch the idea: For a function u on $\mathbb{R}^n$, one may use Fourier transform to go from differentiation to multiplication. Bearing this in mind, one defines

$$\|u\|_s^2 = \int (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi.$$

One then obtains that there are constants C_1, C_2 such that

$$C_1 \|u\|_s^2 \leq \sum_{|\alpha| \leq s} \int |D^\alpha u(x)|^2 dx \leq C_2 \|u\|_s^2,$$

where $D_j = i\frac{\partial}{\partial x_j}$ and we use the usual multi-index notation. Using suitable coordinates and trivialisations, one may define Sobolev norms on manifolds. For an $s \in \mathbb{N}$, they are equivalent to the ones defined above. The Soblev spaces $L^2_s(E)$ are again the completion of $\Gamma(E)$ with respect to the Sobolev norms $\|\cdot\|_s$. For the details, we refer the reader to [LM89, section 8.2].

Remark. Using suitable machinery, one can define real powers of differential operators and define the above norms via

$$\|u\|_s = \|(1 + \Delta)^{\frac{s}{2}} u\|_{L^2}.$$

We now collect some properties of the Sobolev spaces.

Proposition A.3.1 ([LM89, section III.2]). *Let M be a closed Riemannian manifold of dimension n. Then,*

1. *The pairing on $\Gamma(E) \times \Gamma(E^*)$ given by $\langle u, \alpha \rangle = \int_M \alpha(u) dM$ induces a a prefect pairing between $L^2_s(E)$ and $L^2_{-s}(E^*)$, i.e. we have an isomorphism $(L^2_s(E))^* \simeq L^2_{-s}(E^*)$. If E is Hermitian, the L^2-scalar product on $\Gamma(E)$ induces a perfect pairing between $L^2_s(E)$ and $L^2_{-s}(E)$.*

2. *For every k and $s > \frac{n}{2} + k$, there is a continuous inclusion $L^2_s(E) \subset C^k(E)$. Furthermore, any sequence (u_j) that is bounded in $L^2_s(E)$ has a subsequence that converges in $C^k(E)$.*

3. *For $s > t$, we have $L^2_s(E) \subset L^2_t(E)$ and the inclusion is compact.*

4. *A differential operator $P\colon \Gamma(E) \to \Gamma(E)$ of (usual) order k extends to a continuous linear operator $P\colon L^2_s(E) \to L^2_{s-k}(E)$ for any $s \geq k$.*

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